LINEAR SEMI-INFINITE PROGRAMMING: TRENDS AND CHALLENGES

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OUTLINE

Fundamentals of LSIP:

- Linear semi-infinite programs
- Main families of LSIP problems
- The existence question

LSIP applications

LSIP methods

Stability in LSIP

Remark
This talk is based on the paper “Linear semi-infinite optimization: recent advances”, in *Continuous Optimization: Current Trends and Applications* (A. Rubinov and V. Jeyakumar, eds.), Kluwer, to appear.

Linear semi-infinite optimization: recent advances

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EURO XX, Rodhes, July 2004

LINEAR SEMI INFINITE PROGRAMS

We associate with a given triple $\pi = (c, a, b) \in \mathbb{R}^n \times (\mathbb{R}^n)^T \times \mathbb{R}^T$, the LSIP problems

$$(P) \quad \text{Inf } c'x \text{, s.t. } a_t'x \geq b_t, \quad \text{for all } t \in T,$$

and

$$(D) \quad \text{Sup } \sum_{t \in T} \lambda_t b_t \text{, s.t. } \sum_{t \varepsilon T} \lambda_t a_t = c, \quad \lambda \in \mathbb{R}_+^{(T)},$$

where $\mathbb{R}_+^{(T)}$ is the positive cone in $\mathbb{R}^{(T)}$ (the space of generalized finite sequences). It is easy to prove the weak duality theorem: $v(D) \leq v(P)$.

Notation

σ is the constraints system of (P),
F is the feasible set of (P),
B is the boundary of F,
E the set of extreme points of F,
and
F^* is the optimal set of (P).

Most of the information on $\sigma = \{a_t' x \geq b_t, t \in T\}$ is captured by its *characteristic cone*,

$$K = \operatorname{cone}\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

Many families of closed convex sets have been characterized by means of their *reference cone*, $\operatorname{cl} K$ (which does not depend on the chosen linear representation), in Goberna, Jornet & Rodriguez, Contr. to Algebra and Geometry 43 (2002), 153-169.

We associate with $x \in F$ the *cone of feasible directions*

$$D(F;x) = \{d \in \mathbb{R}^n \mid \exists \theta > 0, x + \theta d \in F\}$$

and the *active cone*

$$A(x) := \operatorname{cone}\{a_t,\ t \in T(x)\},$$

where $T(x) = \{t \in T \mid a_t' x = b_t\}$ (set of active indices at x).

MAIN FAMILIES OF LSIP PROBLEMS

σ (or π) is *continuous* (*analytic, polynomial*) if T is a compact Hausdorff space (a compact interval, resp.) and the coefficients are continuous (analytic, polynomial, resp.) on T. Obviously,

σ polynomial $\Longrightarrow$ σ analytic $\Longrightarrow$ σ continuous

σ is FM if every consequence of σ is consequence of a finite subsystem (i.e., K is closed).

σ is LOP if $D\left(F;x\right) = A\left(x\right)^0$ $\forall x \in F$.

σ is LFM if every consequence of σ binding at a certain point of F is consequence of a finite subsystem (i.e., $D\left(F;x\right)^0 = A\left(x\right)$ $\forall x \in F$).

σ cont. & Slater $\Longrightarrow$ σ FM $\Longrightarrow$ σ LFM $\Longleftarrow$ σ LOP

◆ **Duality**: σ is FM and (D) is consistent, then $v(D) = v(P)$ and (D) is solvable.

◆ **Optimality**: Given $x \in F$, if $c \in A\left(x\right)$ (KKT condition) then $x \in F^*$, and the converse is true if σ is LFM. If $c \in \operatorname{int} A\left(x\right)$, then $F^* = \{x\}$.

◆ **Viability of the numerical approaches**

Discretization: σ continuous.

Reduction: σ analytic and LFM.

Primal simplex: σ LOP.

Purification and feasible directions: σ analytic or LOP.

Example 1: Find a best L_1 approximation to f from above in $\operatorname{span}\{v_1, ..., v_n\}$, $f, v_1, ..., v_n : [a, b] \mapsto \mathbb{R}$. Since,

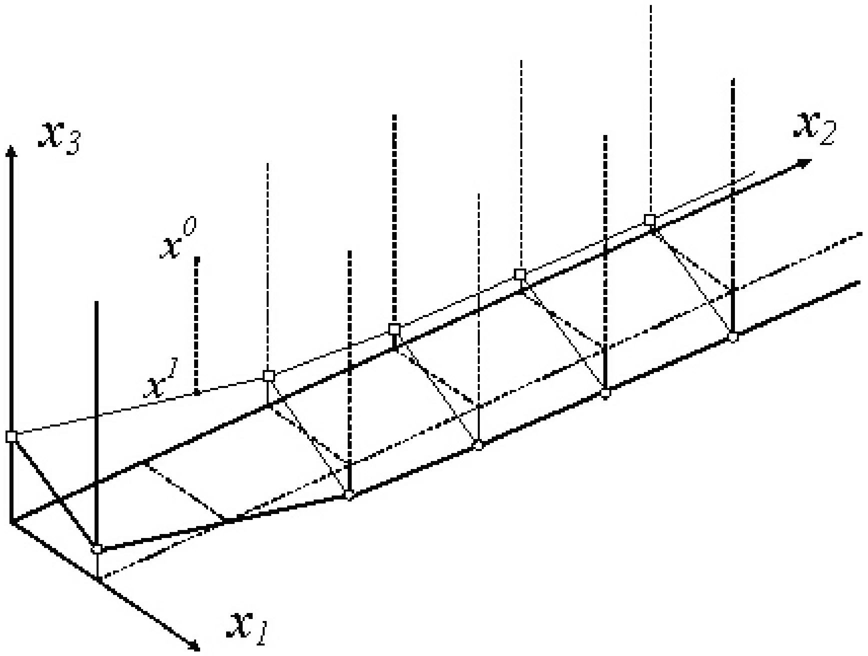
$$\|f - \textstyle\sum_{i=1}^{n} x_i v_i\|_1 = \int_a^b \left[\sum_{i=1}^{n} v_i(t) x_i - f(t)\right] dt$$

$$= \textstyle\sum_{i=1}^{n} \left(\int_a^b v_i(t)dt\right) x_i - \int_a^b f(t)dt,$$

the problem is

$$\begin{array}{ll} (P) & \text{Inf } c'x \\ & \quad \text{s.t } \sum_{i=1}^{n} v_i(t) x_i \geq f(t),\ t \in [a, b], \end{array}$$

where $c_i = \int_a^b v_i(t)dt, i = 1, ..., n$. Here σ is continuous (analytic, polynomial) if $f, v_1, ..., v_n$ are continuous (analytic, polynomial, respectively).



1.

Example 2: The following system is LOP:

$$\left\{ \begin{array}{c} t\left(t+1\right)x_1+x_2+t\left(t+1\right)x_3\geq 2t+1, t\in\mathbb{N} \\ 1\geq x_1\geq 0, x_2\geq 0 \end{array} \right\}$$

THE EXISTENCE QUESTION

Problem: given a family of consistent LSISs, determine the corresponding class of solution sets.

Let $F \neq \emptyset$ be a closed convex set.

◆ F always admits continuous, FM, and LFM external representation: if $T_1 := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^n \mid a'x \geq b \, \forall x \in F \right\}$, $T_2 := \{t \in T_1 \mid \|t\| \leq 1\}$, and

$$\sigma_i := \left\{ a'x \geq b, \begin{pmatrix} a \\ b \end{pmatrix} \in T_i \right\}, i = 1, 2,$$

then σ_1 is FM (LFM) and σ_2 is continuous.

◆ F admits LOP representation if and only if F is quasipolyhedral (i.e., the nonempty intersections of F with polytopes are polytopes).

◆ F does not admit analytic representation if one of the following conditions hold (Jaume & Puente, *LAA*, to appear):

(i) F is a quasi-polyhedral non-polyhedral set.

(ii) $n \geq 3$, F is smooth (i.e., $\exists$ a unique supporting half-space $\forall x \in \operatorname{bd} F$), and $\dim \operatorname{lin} F \leq n-3$ (e.g., F is bounded).

♦ Sufficient conditions for F to admit polynomial representation can be found in Goberna, Hernández & Todorov, *JOTA*, to appear. For instance,

F	$\deg F$
$\{x \in \mathbb{R}^n \mid c_i' x \geq d_i, i = 1, ..., p\}$	$\max\{0, 2p-3\}$
convex hull of an ellipse	4
convex hull of a parabola	4
convex hull of a branch of hyperbola	2

♦ Separation: The solution set of the analytic system $\{te^t x_1 + tx_2 \leq 1, t \in [0,1]\}$ has no polynomial representation.

RECENT LSIP APPLICATIONS

Economics

◆ Kortanek & Medvedev, *Building and Using Dynamic Interest Rate Models*, Wiley (2001)

Dynamic interest rate models formulated as FM-analytical LSIP problems; two-phase method.

◆ Konno & Kobayashi, *Asia-Pacific Financial Markets* 7 (2000) 261-273

Failure discrimination and rating of enterprises (separation by convex quadratic surfaces). A continuous LSIP problem is solved via cutting-plane discretization.

◆ Gretsky, Ostroy & Zame, *Positivity* 6 (2002) 261-264

Continuous assignment problem of mathematical economics; LSIP duality theory.

◆ Jerez, *J. Econ. Theory* 112 (2003) 1-34

Characterization of efficient allocations under asymmetric information; LSIP duality theory.

Game Theory

◆ Marinacci & Montrucchio, *J. Mathematical Economics* 39 (2003) 1-25

Transferable utility games (cooperative game theory); calculus of the linear core through a primal LSIP problem.

◆ Sanchez-Soriano, Llorca, Tijs & Timmer, *SIP: Recent Advances*, Kluwer (2001) 349-363

Semi-infinite transportation problems; LSIP duality theory.

◆ Tijs, Timmer, Llorca & Sanchez-Soriano, *SIP: Recent Advances*, Kluwer (2001) 365-386

Linear production situations with an infinite number of production techniques; primal and dual games associated to a certain LSIP problem.

Probability and Statistics

◆ Dall'Aglio, *SIP: Recent Advances*, Kluwer (2001) 237-254

Subjective probability theory, maximum likelihood estimation and risk theory; LSIP duality theory and methods.

◆ Betró & Guglielmi, *Robust Bayesian Analysis*, Springer (2000) 273-293

Bayesian robustness; the optimization of posterior functionals over a generalized moment class as a dual LSIP.

◆ Noubiap & Seidel, *The Annals of Statistics* 29 (2001) 1094-1116

Bayesian statistics; the calculus of minimax decision rules under generalized moment conditions; if the space of parameters (where the family of distributions is defined) is compact, then the convergence test consists of solving a certain continuous LSIP problem; LSIP discretization algorithm.

Geometry

♦ Goberna, López & Wu, *SIP: Recent Advances*, Kluwer (2001), 255-269

Separation and strong separation in normed spaces by means of LSIP.

♦ Juhnke & Sarges, *Contr. Algebra & Geometry* 41 (2000), 93-105

Characterization of the minimal shell of a convex body; LSIP duality theory.

Machine Learning

♦ Rätsch, Demiriz & Bennet, *Machine Learning* 48 (2002), 189-218

The generation of sequences of functions (hypotheses) from an infinite class of functions is formulated as LSIP problem.

Data envelopment analysis

♦ Jess et al., *Optimization* 49 (2001) 369-385

Comparison of LSIP and bilevel optimization approaches to DEA models with infinitely many DMUs. A numerical example is provided.

Telecommunication networks

◆ Nordholm et al., *Annals of Oper. Res.* 98 (2001) 235-253

Improving the capacity of mobile networks by filtering the signal through a beamforming structure; optimal design of this structure by solving an analytic LSIP problem; numerical experiments with a simplex-like method.

◆ Dahl et al., *Optimization Methods and Applications*, Kluwer (2001)3 09-319

Design of narrow-band antennas; similar LSIP problems and numerical treatment.

◆ Sabharwal, Avidor & Potter, *IEEE Trans. on Vehicular Tech.* 49 (2000) 1784-1792

Improving the capacity of cellular systems through cell sectorization; a certain continuous LSIP problem solves a technical difficulty; numerical experiments with a discretization procedure.

Control problems

◆ Rubio, *J. Franklin Inst.* 337 (2000), 673-690

Optimal boundary control problem associated to a certain nonlinear diffusion equation with a “rough” initial condition; continuous dual LSIP problem.

◆ Rubio, *Optimization* 48 (2000), 191-210

Two kinds of optimal control problems with unbounded control sets are formulated as continuous dual LSIP problems.

◆ Slupphaug, Imsland & Foss, *Modeling, Identification and Control* 22 (2001) 29-52

Robust control of nonlinear systems with uncertain parameters; continuous LSIP problems; numerical experiments with a discretization method.

Optimization under uncertainty

♦ Amaya & Gómez, *Optimization* 49 (2001) 243-369

The paper provides strong duality theorems for inexact LP problems; under certain hypotheses, they become continuous dual LSIP problems.

♦ Hu & Fang, *Comput. Math. Appl.* 40 (2000) 721-733

Fuzzy linear semi-infinite systems formulated as continuous LSIP (and NLSIP) problems; cutting plane approach; numerical experiments are provided.

♦ León & Vercher, *SIP: Recent Advances*, Kluwer (2001), 327-348

Set inclusive constraint models and fuzzy programming problems; LSIP problems with analytic constraint subsystems and box constraints; examples solved with a hybrid method.

Semidefinite programming

♦ Kortanek & Zhang, *Math. Programming* 91 (2001) 127-144

Comparing SDP and LSIP duality theories.

♦ Konno et al., *J.Global Optim.* 25 (2003) 141-155 and *J. Comput. and Appl. Math.*, to appear

The dual problem in SDP is reformulated as a continuous (P); cutting-plane method.

♦ Krishnan Mitchel, *Novel Approaches to Hard Discrete Optimization*, AMS (2003) 121-140

Similar approach to SDP; if (P) is FM and $v\,(P) \neq -\infty$, then it is replaced with a small discretization; numerical experiments with large scale SDPs.

RECENT LSIP METHODS

Discretization methods

♦ Betró, *Math. Program*, to appear

An accelerated Elzinga-Moore central cutting plane algorithm for continuous LSIP; numerical experiments are reported.

♦ Wu, Fang & Lin, *SIP: Recent Advances*, Kluwer (2001), 221-233

An analytic center cutting plane method for analytic LSIP problems.

♦ Fang, Lin & Wu, *J. Comput. Appl. Math.* 129 (2001) 89-104

A relaxed cutting-plane scheme for LSIP and QSIP.

♦ Konno, Gotho & Uno, *J. Comput. and Appl. Math.*, to appear

A Kelley cutting-plane algorithm for an important class of LSIP problems (the reformulations of dual SDP problems).

Simplex like method

♦ Anderson, Goberna & López, *MOR* (2001), 147-162

A simplex method and a reduced gradient method for LOP LSIP problems.

Hybrid method

♦ León, Sanmatías & Vercher, *EJOR* 121 (2000) 78-91

Two methods for LSIP problems with blocks of analytic systems and box constraints; hybrid numerical approach; numerical experiments are reported.

Reduction method

♦ Ito, Liu & Teo, *Annals of Oper. Res*. 98 (2000) 189-213

A reduction method (based on Wolfe's duality) for FM-continuous LSIP problems; good performance on a famous test example.

Other approaches

◆ Kosmol & Müller-Wichards, *J. Contemp. Math. Anal.* 36 (2002) 31–48

An adapted Polya approximation method for a class of FM-continuous LSIP problems.

◆ Kostyukova, *JOTA* 110 (2001) 585-609

A path-following algorithm for (P_τ), $\tau \in [0, \overline{\tau}]$, where (P_τ) is the FM-continuous LSIP problem resulting of replacing T with an interval $T(\tau) := [0, \tau]$, with $\tau \in [0, \overline{\tau}]$, $\overline{\tau} > 0$; an illustrative example is solved.

STABILITY IN LSIP 2000-2003

We consider arbitrary perturbations of the nominal data $\pi = (c, a, b)$ which preserve n and T. So the *parameters space* is $\Pi := \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R})^T$, endowed with the pseudometric:

$$d(\pi_1, \pi) = \max \left\{ \left\| c^1 - c \right\|, \sup_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \right.$$

where $\pi_1 = (c^1, a^1, b^1)$.

The **problems** associated with π_1 are (P_1) and (D_1).

Set-valued mappings: $\mathcal{F}(\pi_1) = F_1$, $\mathcal{B}(\pi_1) = B_1$, $\mathcal{E}(\pi_1) = E_1$, and $\mathcal{F}^*(\pi_1) = F_1^*$.

Ordinary mapping: $\vartheta(\pi_1) = v(P_1)$.

Sets: $\Pi_c = \{\pi_1 \in \Pi \mid F_1 \neq \emptyset\}$, $\Pi_i = \{\pi_1 \in \Pi \mid F_1 = \emptyset\}$, $\Pi_b = \{\pi_1 \in \Pi \mid v(P_1) \neq -\infty\}$, and $\Pi_s = \{\pi_1 \in \Pi \mid F_1^* \neq \emptyset\}$.

STABILITY OF $\mathcal{F}$

⧫ **Closedness**: $\mathcal{F}$ is closed.

⧫ **Berge lower semicontinuity (lsc)**: Each of the following conditions is equivalent to $\mathcal{F}$ being lsc at $\pi \in \Pi_c$ (Gobena, López & Todorov 1996-1998):

♢ $\pi \in \operatorname{int} \Pi_c$.

♢ Sufficiently small perturbations of b preserve the consistency.

♢ $\exists \overline{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $a_t' \overline{x} \geq b_t + \varepsilon$ for all $t \in T$ (SS cond.).

♢ $0_{n+1} \notin \operatorname{cl} \operatorname{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\}$.

♢ $\forall \{\pi_r\} \subset \Pi$ such that $\lim\limits_{r\to\infty} \pi_r = \pi$ $\exists r_0 \in \mathbb{N}$ such that $\lim_{r \geq r_0} F_r = F$.

♢ $\exists$ an open set V, $\pi \in V \subset \Pi$, such that $\dim F_1 = \dim F$ for all $\pi_1 \in V$.

♢ $\exists$ an open set V, $\pi \in V \subset \Pi$, such that $F_1 \approx F$ for all $\pi_1 \in V$ (provided that $0_n \notin \operatorname{bd} \operatorname{conv} \{a_t, t \in T\}$).

◆ **Berge upper semicontinuity (usc)**:
Let K^R be the characteristic cone of

$$\sigma^R = \left\{ a'x \geq b, \begin{pmatrix} a \\ b \end{pmatrix} \in \left(\operatorname{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\} \right)_\infty \right\},$$

where $X_\infty = \left\{ \lim_k \mu_k x^k \mid \left\{ x^k \right\} \subset X, \{\mu_k\} \downarrow 0 \right\}$. If F is bounded, then $\mathcal{F}$ is usc at π. Otherwise (Cánovas, López & Parra, *SVA 10 (2002)*, 361-378):

◇ If F contains at least one line, then $\mathcal{F}$ is usc at $\pi \iff K^R = \operatorname{cl} K$.

◇ Otherwise, if w is the sum of a certain basis of $\mathbb{R}^n$ contained in $\{a_t, t \in T\}$, then $\mathcal{F}$ is usc at π iff $\exists \beta \in \mathbb{R}$ such that

$$\operatorname{cone} \left(K^R \cup \left\{ \begin{pmatrix} w \\ \beta \end{pmatrix} \right\} \right) = \operatorname{cone} \left(\operatorname{cl} K \cup \left\{ \begin{pmatrix} w \\ \beta \end{pmatrix} \right\} \right).$$

Dual counterparts for these results can be found in Goberna, López & Todorov, *SVA* 9 (2001) 75-99.

STABILITY OF $\mathcal{B}$

Given $\pi \in \Pi_c$ such that $F \neq \mathbb{R}^n$, then we have (Goberna, Larriqueta & Vera, *SVA* 11 (2003) 203-223):

$$\begin{tikzcd}
\mathcal{F} \text{ lsc at } \pi \arrow[r, leftrightarrow] & \mathcal{B} \text{ lsc at } \pi \arrow[dr, "(1)"] & \\
& & \mathcal{B} \text{ closed at } \pi \\
\mathcal{F} \text{ usc at } \pi & \mathcal{B} \text{ usc at } \pi \arrow[l] \arrow[ur, "(2)"'] &
\end{tikzcd}$$

Remarks

(1) The converse holds if $\dim F = n$.

(2) The converse holds if F is bounded.

STABILITY OF $\mathcal{E}$

σ is *nondegenerate* if

$$|T(x)| < n \text{ for all } x \in B \backslash E.$$

Related conditions:

$$\begin{array}{c} \{a_t, t \in T(x)\} \text{ lin. ind. } \forall x \in B \backslash E \text{ s.t. } T(x) \neq \emptyset \\ \downarrow^{(1)} \\ \sigma \text{ nondegenerate} \\ \downarrow^{(2)} \\ \left|T\left(\frac{x+y}{2}\right)\right| < n \; \forall x, y \in E, \text{ such that } x \neq y \end{array}$$

Remarks

(1) The converse holds if F is bounded and $|T| < \infty$.

(2) The converse holds if F is bounded.

Let $\sigma_H = \{a_t'x \geq 0, t \in T\}$. If $|T| \geq n$, $E \neq \emptyset$, and $|F| > 1$, then we have (Goberna, Larriqueta & Vera, submitted):

$$\begin{tikzcd}
\mathcal{F} \text{ lsc at } \pi \arrow[r, leftrightarrow] & \mathcal{E} \text{ lsc at } \pi \arrow[d, "(1)"] & \\
& \mathcal{E} \text{ closed at } \pi \arrow[r, "(4)"] \arrow[d, shift right, "(2)"'] & \sigma \text{ nondeg} \\
& \mathcal{E} \text{ usc at } \pi \arrow[r] \arrow[u, shift right, "(3)"'] & \sigma \ \& \ \sigma_H \text{ nondeg}
\end{tikzcd}$$

Remarks

(1) If F is strictly convex.

(2) If F is bounded.

(3) If $\{a_t, t \in T\}$ is bounded.

(4) If $\mathcal{F}$ is lsc at π.

Moreover, if $|T| < \infty$ then

$$\begin{tikzcd}
\mathcal{E} \text{ usc at } \pi \arrow[r, leftrightarrow] \arrow[d, "(1)"] & \sigma \text{ and } \sigma_H \text{ nondegenerate} \\
\mathcal{E} \text{ closed at } \pi \arrow[r, leftrightarrow] \arrow[d] & \sigma \text{ nondegenerate} \\
\mathcal{E} \text{ lsc at } \pi \arrow[r, leftrightarrow] & \mathcal{F} \text{ lsc at } \pi
\end{tikzcd}$$

Remark

(1) If F is bounded.

STABILITY OF $\mathcal{F}^*$ AND ϑ, AND WELL-POSEDNESS

The next results are due to Cánovas, López, Parra & Todorov (1999).

Let $\pi \in \Pi_s$.

◆ $\mathcal{F}^*$ is closed at $\pi \longleftrightarrow$ either $\mathcal{F}$ is lsc at π or $F = F^*$.

◆ $\mathcal{F}^*$ is lsc at $\pi \longleftrightarrow \mathcal{F}$ is lsc at π and $|F^*| = 1$.

◆ If $\mathcal{F}^*$ is usc at π, then $\mathcal{F}^*$ is closed at π (and the converse is true if F^* is bounded).

Let $\pi \in \Pi_c$.

◆ If $F^* \neq \emptyset$ and bounded, then ϑ is lsc at π (and the converse is true if $\pi \in \Pi_b$).

◆ ϑ is usc at $\pi \longleftrightarrow \mathcal{F}$ is lsc at π.

$\{x^r\} \subset \mathbb{R}^n$ is an *asymptotically minimazing sequence* (a.m.s) for π associated with $\{\pi_r\} \subset \Pi_b$ if $x^r \in F_r$ $\forall r$, $\lim_r \pi_r = \pi$, and $\lim_r \left[(c^r)' x^r - v(P_r)\right] = 0$.

Given $\pi \in \Pi_s$, π is *Hadamard well-posed* (Hwp) if $\forall x^* \in F^*$ and $\forall \{\pi_r\} \subset \Pi_b$ such that $\lim_r \pi_r = \pi$ there exists an a.m.s. converging to x^*.

◆ If π is Hwp, then $\vartheta|_{\Pi_b}$ is continuous.

◆ If F^* is bounded, π is Hwp $\longleftrightarrow$ either $\mathcal{F}$ is lsc at π or $|F| = 1$.

◆ If F^* is unbounded and π is Hwp, then $\mathcal{F}$ is lsc at π.

Other variants of Hwp are characterized in Cánovas, López, Parra & Todorov, Annals Oper. Res. 101 (2001) 171-190.

DISTANCE TO ILL-POSEDNESS

The next results are due to Cánovas, López, Parra & Toledo, submitted.

♦ $\operatorname{bd}\Pi_c$ is the set of *ill-posed problems in the feasibility sense*.

If $\pi \in \Pi_c$, $d(\pi, \operatorname{bd}\Pi_c) = |\varkappa|$, where

$$\varkappa = \sup_{x\in\mathbb{R}^n} \inf_{t\in T} \left\| \begin{pmatrix} x \\ -1 \end{pmatrix} \right\|_*^{-1} (a_t' x - b_t)$$

is the consistency value of π. Moreover,

$$\begin{cases} \pi \in \operatorname{int}\Pi_c & \longleftrightarrow & \varkappa > 0 \\ \pi \in \operatorname{bd}\Pi_c & \longleftrightarrow & \varkappa = 0 \\ \pi \in \operatorname{int}\Pi_i & \longleftrightarrow & \varkappa < 0 \end{cases}$$

Let Π_{si} be the set of problems which have a finite inconsistent subproblem.

♦ $\operatorname{bd}\Pi_{si}$ is the set of *generalized ill-posed problems in the feasibility sense*.

If $\pi \in \Pi_c$, then

$$d(\pi, \operatorname{bd}\Pi_{si}) = d(0_{n+1}, \operatorname{bd} H),$$

where

$$H := \operatorname{conv}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T\right\} + \operatorname{cone}\left\{\begin{pmatrix} 0_n \\ -1 \end{pmatrix}, t \in T\right\}$$

is the so-called *hypographical set* of π.

◆ $\operatorname{bd} \Pi_s = \operatorname{bd} \Pi_b$ is the set of *ill-posed problems in the optimality sense*.

Let $\pi \in \operatorname{int} \Pi_c$.

Then $\pi \in \operatorname{bd} \Pi_s \longleftrightarrow 0_n \in \operatorname{bd} Z^-$,with $Z^- := \operatorname{conv}\{a_t, t \in T; -c\}$.

Moreover for $\pi \in (\operatorname{cl} \Pi_s) \cap (\operatorname{int} \Pi_c)$,

$$d(\pi, \operatorname{bd} \Pi_s) = \min\{d(0_{n+1}, \operatorname{bd} H), d(0_n, \operatorname{bd} Z^-)\}.$$

Let $\pi \in \operatorname{bd} \Pi_c$.

Then $\pi \in \operatorname{bd} \Pi_s \longleftrightarrow 0_n \in \operatorname{bd} Z^+$, where $Z^+ := \operatorname{conv}\{a_t, t \in T; c\}$.

Moreover for $\pi \in (\operatorname{cl} \Pi_s) \cap (\operatorname{bd} \Pi_c)$,

$$d(\pi, \operatorname{bd} \Pi_s) \geq \min\{d(0_{n+1}, \operatorname{bd} H), d(0_n, \operatorname{bd} Z^-)\}.$$

ERROR BOUNDS

The *residual funtion* of π is

$$r(x,\pi) := \sup_{t\in T} (b_t - a_t' x)^+ ,$$

where $\alpha^+ := \max\{\alpha, 0\}$. Obviously, $x \in F \Longleftrightarrow r(x,\pi) = 0$.

The following (Robinson's) condition is equivalent to $\mathcal{F}$ is lsc at π: $\forall x \in F\ \exists \beta_x > 0$ and an open set V_x, $\pi \in V_x \subset \Pi$, such that either $x \in F_1$ or $\dfrac{d(x,F_1)}{r(x,\pi_1)} \leq \beta_x$, and this $\forall \pi_1 \in V_x$.

$0 \leq \beta < +\infty$ is a *global error bound* for $\pi \in \Pi_c$ if

$$\frac{d(x,F)}{r(x,\pi)} \leq \beta\ \forall x \in \mathbb{R}^n \backslash F.$$

If there exists such a β, then the condition No. of π is

$$0 \leq \tau := \sup_{x\in\mathbb{R}^n\backslash F} \frac{d(x,F)}{r(x,\pi)} < +\infty$$

Hu, *Math. Programming (Ser. B)* 88 (2000) 277-284) has given upper bounds for τ

in the proximity of π: If $\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\}$ is bounded, then:

(i) Assume that F is bounded and $\pi \in \operatorname{int} \Pi_c$, and let ρ, x^0 and $\varepsilon > 0$ such that $\|x\| \leq \rho \, \forall x \in F$ and $a_t' x^0 \geq b_t + \varepsilon \, \forall t \in T$. Let $0 \leq \gamma < 1$. Then, if $d(\pi_1, \pi) < \dfrac{\varepsilon \gamma n^{-\frac{1}{2}}}{2\rho}$, we have

$$\tau_1 \leq 2\rho\varepsilon^{-1} \left[\frac{1+\gamma}{(1-\gamma)^2} \right].$$

(ii) Assume that F is unbounded and $\sigma_H \in \operatorname{int} \Pi_c$, and let u and $\eta > 0$ such that $a_t' u \geq \eta$ $\forall t \in T$, $\|u\| = 1$. Let $0 < \delta < n^{-\frac{1}{2}} \eta$. Then, if $d(\pi_1, \pi) < \delta$, we have

$$\tau_1 \leq \left(\eta - \delta n^{\frac{1}{2}} \right)^{-1}.$$

Improved error bounds have been obtained by Cánovas, López, Parra & Toledo, submitted, where it is proved that $\mathcal{F}^{-1}$ is metrically regular at any point $(z, \pi) \in \operatorname{gph} \mathcal{F}^{-1}$ such that $\pi \in \operatorname{int} \Pi_c$.