Control and optimal response problems for quasilinear impulsive integrodifferential equations

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Abstract

One of the central results of the paper is the Pontryagin maximum principle [16] which is considered in sufficient form for the linear case of impulsive differential equations. The problem of controllability of boundary-value problems for quasilinear mpulsive system of integrodifferential equations is considered. The control consists of a piecewise continuous function part as well as impulses which act at a variable time.

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1 Introduction

The theory of impulsive differential equations is emerging as an important area of investigation since it is richer in problems in comparison with the corresponding theory of differential equations. Actually, many mathematical, e.g., dynamical and optimization problems encountered in studying impulsive differential equations cannot be treated with the usual techniques of ordinary differential equations [11, 14, 17]. Here, we also mention biological applications in population dynamics and genetics (mutation, experiments, etc., [21]) where impulses (jumps) naturally arise or are caused by control. Moreover, impulsive differential systems represent a natural framework for mathematical modeling of several real-world problems [7, 9, 11, 14, 17]. However, the theory of integrodifferential equations with impulse actions on surfaces is not yet sufficiently elaborated compared to that of impulsive differential equations

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and integrodifferential equations. There are also several problems on the controllability problem for impulsive systems which are connected with the results of the theory of integral and integrodifferential equations and have not been well investigated yet [2, 5, 6, 8, 10, 12, 15, 18]. One of the original approaches to optimization problems is throgh boundary value problem. And this method request correspondly the development of the controllability problem of solutions for impulsive equations, particularly of integrodifferential systems of the form

$$\begin{aligned}
&dx/dt = A(t)x + \int_\alpha^t K(t,s)x(s)ds + C(t)u(t) + f(t) + \mu g(t,x,u,\mu), \quad t \neq \theta_i + \mu\tau_i(x,\mu),\\
&\Delta x(\zeta_i) = B_i x(\zeta_i) + \sum_{j:\alpha<\zeta_j\le\zeta_i} D_{ij}x(\zeta_j) + \int_\alpha^{\zeta_i} M_i(s)x(s)\,ds + Q_i v_i + J_i + \mu W_i(x(\zeta_i), v_i, \mu),\\
&i = 1, 2, \dots, p, \qquad (1)\\
&x(\alpha) = a, \quad x(\beta) = b. \qquad (2)
\end{aligned}$$

Here, $\mu > 0$ is a small parameter, $\alpha, \beta, \theta_i, \zeta_i \in R$ such that $\alpha < \theta_1 < \cdots < \theta_p < \beta$, and $\zeta_i = \theta_i + \mu\tau_i(x(\zeta_i), \mu)$; A, K, M_i, D_{ij} and B_i are $n \times n$ matrices; C and Q_i are $n \times m$ matrices; x, f, g, J_i, W_i, a, and b are n-vectors; u and v are m-vectors; $\tau_i(x,\mu), i = 1, 2, \dots, p$, are real valued scalar functions; $\Delta x(t) \equiv x(t+) - x(t-)$, where $x(t+) = \lim_{h\to 0^+} x(t+h)$ and $x(t-) = \lim_{h\to 0^-} x(t+h)$. We assume that solutions are left continuous and therefore write $\Delta x(t) = x(t+) - x(t)$. In this paper, by the help of some results from [1, 3, 5], we shall investigate the problem of controllability of boundary-value problems for quasilinear impulsive system of integrodifferential equations of the form (1), (2). We obtain our results by comparing solutions of integrodifferential equations having impulse actions at variable moments with solutions of integrodifferential equations having impulse actions at fixed moments. This comparison method was proposed by M.U. Akhmetov and N.A. Perestyuk in [1]. As being well-known, the solutions of differential equations with variable moments of impulse effect may experience pulse phenomena, namely, they may hit a given surface of discontinuity a finite or infinite number of times causing rhythmical beating [14, 17]. This results in additional complications in studying such systems and, therefore, in most cases it is necessary to find conditions that guarantee the absence of beating. In the present article we also give a new condition for the absence of beating (see Theorem 5) which is based on the method of small parameter. Finally, a maximum principle of Pontryagin type is proposed for the time-optimal problem in linear case. The results of this paper may be considered as a continuation or a generalization of the results obtained in [3], [4] and [5], where linear and quasilinear impulsive differential systems were considered. The results can be useful to investigate problems of the optimum control for discontinuous dynamics [7, 16].

2 Preliminaries

In the following, by $PAC[\alpha,\beta]$ we denote the set of all functions $x : [\alpha,\beta] \to R^n$ being piecewise absolutely continuous and continuous on the left with discontinuities of the first kind at points $\theta_i, i = 1, 2, \dots, p$. By $L_2^r[\alpha,\beta]$ we denote the set of all square integrable functions $\phi : [\alpha,\beta] \to R^r$ and by $D^r[1,p]$ the set of all finite sequences $\{\xi_i\}, \xi_i \in R^r$, $i = 1, \dots, p$, where p and r are fixed positive integers. Furthermore, we define $\Pi^r[\alpha,\beta] := L_2^r[\alpha,\beta] \times D^r[1,p]$ and identify its elements as $\{\phi, \xi\}$, and let

$$\big\langle \{\phi,\xi\},\{\omega,\nu\}\big\rangle = \int_\alpha^\beta (\phi,\omega)\,dt + \sum_{i=1}^{p} (\xi_i,\nu_i)$$

be an inner product in $\Pi^r[\alpha,\beta]$, where $(,)$ is the Euclidean scalar product in R^r. Let us introduce the norm $||\{\phi,\xi\}||_{[\alpha,\beta]} = \big\langle \{\phi,\xi\},\{\phi,\xi\}\big\rangle^{\frac{1}{2}}$ in $\Pi^r[\alpha,\beta]$. Throughout this paper we need the following conditions:

(C1) the functions g, W_i, τ_i, $i = 1, 2, \dots, p$, are continuous with respect to their variables and continuously differentiable with respect to x, u, and v;

(C2) the matrix $K(t,s) : [\alpha,\beta] \times [\alpha,\beta] \to R^n \times R^n$ is square integrable;

(C3) the columns of the matrices $A(t)$ and $M_i(t)$, $i = 1, 2, \dots, p$, are in $L_2^n[\alpha,\beta]$;

(C4) $\{f, J\} \in \Pi^n[\alpha,\beta]$;

(C5) $det(I + B_j + D_{jj}) \neq 0, det(I + B_j) \neq 0$ for $j = 1, 2, \dots, p$.

Definition 2.1 *Problem (1), (2), which we denote by $\Sigma_\mu(G)$, is said to be solvable for given bounded set $G = G_a \times G_b \subset R^n \times R^n$ if there exists a positive real number μ_0, $\mu_0 = \mu_0(G)$, such that for all arbitrary $a, b \in G_a \times G_b$ and $\mu < \mu_0$ there is a control $\{u, v\} \in \Pi^m$ for which system (1) admits a solution $x(t)$ satisfying (2).*

Let s be a positive real number, and let T_s be the subset of elements (x, u, v) satisfying the inequality $|x| + |u| + |v| \leq s$, where $| \cdot |$ is the Euclidean norm in R^n.

For fixed positive real number μ_1, we define

$$G_s = \{(x,u,v,t,i,\mu) : (x,u,v) \in T_s, \;\; \alpha \le t \le \beta, \;\; i = 1,2,\ldots,p, \;\; 0 < \mu \le \mu_1\}.$$

Let a positive real number H be fixed and

$$\begin{aligned}
m_1 &= \max\left\{ \sup_t |A(t)|, \sup_t |C(t)|, \sup_{t,s} |K(t,s)|, \sup_{i,t} |M_i(t)|, \max_i |B_i|, \max_{ij} |D_{ij}| \right\}, \\
m_2 &= \max\left\{ \sup_t |f(t)|, \max_i |J_i| \right\}, \\
m_3 &= \max\left\{ \max_{(t,x,u,\mu)\in G_H} |g|, \max_{(x,v,\mu,i)\in G_H} |W_i|, \max_{(x,\mu,i)\in G_H} |\tau_i| \right\}.
\end{aligned}$$

It is not very difficult to observe in view of (C1) that there is a positive real number L such that

$$\begin{aligned}
&|g(t,x_1,u_1,v^1,\mu) - g(t,x_2,u_2,v^2,\mu)| \le L\left\{|x_1 - x_2| + |u_1 - u_2| + |v^1 - v^2|\right\}, \\
&|W_i(x_1,v^1,\mu) - W_i(x_2,v^2,\mu)| \le L\left\{|x_1 - x_2| + |v^1 - v^2|\right\}, \\
&|\tau_i(x_1,\mu) - \tau_i(x_2,\mu)| \le L\,|x_1 - x_2|.
\end{aligned}$$

uniformly for all $t, x_1, x_2, u_1, u_2, v^1, v^2$ in G_H,

Definition 2.2 *If for $h > 0$ there exists a positive real number μ_0, $\mu_0 = \mu_0(h)$, such that if $\mu < \mu_0$ then for every given subset $G = \{(a,b) | |a| < h, |b| < h\} \subset R^n \times R^n$ the problem $\Sigma_\mu(G)$ is solvable, then we say that the problem $\Sigma_\mu(G)$ is solvable.*

Lemma 1 *Let D_{ij}, $i,j = 1,2,\ldots,p$, be constant matrices of size $n \times n$ and $\{\xi_i\} \in D^n[1,p]$. Then*

$$\sum_{i:\alpha<\theta_i<t}\;\sum_{j:\alpha<\theta_j\le\theta_i} D_{ij}\xi_j = \sum_{i:\alpha<\theta_i<t}\;\sum_{i:\theta_i\le\theta_j<t} D_{ji}\xi_i \quad \text{for each } t \in [\alpha,\beta].$$

Lemma 2 *Let $K(t,s)$ be a matrix of size $n \times n$. If $K(t,s)$ is square integrable with respect to s on $[\alpha,\beta]$ for each fixed $t \in [\alpha,\beta]$, and $\phi_i(t) \in L_2^n[\alpha,\beta]$ for $i = 1,2,\ldots,n$, then*

$$\int_\alpha^t K(t,s) \sum_{i:\alpha<\theta_i<s} \phi_i(s)\,ds = \sum_{i:\alpha<\theta_i<t} \int_{\theta_i}^t K(t,s)\phi_i(s)\,ds \quad \text{for each } t \in [\alpha,\beta]. \tag{3}$$

Let us consider an integral equation

$$x(t) = \int_\alpha^t G(t,s)x(s)ds + \sum_{i:\alpha<\theta_i<t} S_i(t)x(\theta_i) + \sum_{i:\alpha<\theta_i<t} N_i(t)x(\theta_i+) + \sum_{i:\alpha<\theta_i<t} J_i + f(t), \tag{4}$$

under the following conditions:

(H1) the matrix $G(t,s) : [\alpha,\beta] \times [\alpha,\beta] \to R^n \times\ R^n$ is square integrable;

(H2) the columns of matrices $S_i(t)$ and $N_i(t)$ and the function $f(t)$ belong to $PAC[\alpha,\beta]$;

(H3) $\mathrm{d}et(I - N_i(\theta_i+) + N_i(\theta_i)) \neq 0$ for $i = 1, 2, \dots, p$.

Theorem 1 *Let conditions (H1), (H2), and (H3) hold. Then system (4) has a unique solution $x(t) \in PAC[\alpha,\beta]$ which can be represented as*

$$x(t) = \int_\alpha^t P_1(t,s)f(s)\,ds + \sum_{i:\alpha<\theta_i<t} Q_i(t)J_i + \sum_{i:\alpha<\theta_i<t} P_2^i(t)f(\theta_i) + f(t) + \sum_{i:\alpha<\theta_i<t} J_i,$$

where $Q_i(t)$, $P_2^i(t)$, $i = 1, 2, \dots, p$, and $P_1(t,s)$ are certain piecewise continuous function matrices of size $n \times n$.

3 Existence of solutions of integrodifferential equations

We may now state and prove a theorem on existence and uniqueness of solutions of the impulsive system of integrodifferential equations of the form

$$\begin{aligned} dx/dt &= A(t)x + \int_\alpha^t K(t,s)x(s)ds + f(t), \quad t \neq \theta_i, \\ \Delta x(\theta_i) &= B_i x(\theta_i) + \sum_{j:\alpha<\theta_j\le\theta_i} D_{ij}x(\theta_j) + \int_\alpha^{\theta_i} S_i(s)x(s)\,ds + J_i. \end{aligned} \tag{5}$$

Theorem 2 *Let conditions (C2), (C3), and (C4) be satisfied. Then for a given $x_0 \in R^n$ there exists unique solution $x(t) \in PAC[\alpha,\beta]$ of (5) which satisfies $x(\alpha) = x_0$.*

$$\begin{aligned} \partial\lambda(t,s)/\partial s &= -\lambda(t,s)A(s) - \int_s^t \lambda(t,\sigma)K(\sigma,s)d\sigma - \sum_{j:s\le\theta_j<t} \lambda(t,\theta_j)M_i(s), \quad s \neq \theta_i, \quad t \in [\alpha,\beta], \\ \Delta\lambda(t,\theta_i) &= -\lambda(t,\theta_i)B_i(I+B_i)^{-1} - \sum_{j:\theta_i\le\theta_j<t} \lambda(t,\theta_j+)D_{ji}(I+B_i)^{-1}. \end{aligned} \tag{6}$$

where $\lambda \in R^n$ is row vector, A, K, D_{ij}, M_i and B_i are as before, and $\Delta\lambda(t,\theta_i) := \lambda(t,\theta_i+) - \lambda(t,\theta_i)$.

Theorem 3 *Let conditions (C2) - (C5) hold. Then for a given $\lambda_0 \in R^n$ the system (6) has a unique solution $\lambda(t,s)$ such that $\lambda(t,t) = \lambda_0$.*

Now for every $i = 1, \dots, n$, denote $\lambda_i(t,s)$ be the unique solution of (6) such that if $\Lambda(t,s) = \mathrm{col}(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)$, then $\Lambda(t,t) = I$.

Theorem 4 *Let* $x(t) = x(t, \alpha, x_0)$ *be a solution of the Cauchy problem for (5). Then* $x(t)$ *has the representation*

$$x(t) = \Lambda(t, \alpha)x_0 + \int_{\alpha}^{t} \Lambda(t, s)f(s)ds + \sum_{i:\alpha<\theta_i<t} \Lambda(t, \theta_j+)J_i. \tag{7}$$

4 Comparison method for integro differential impulsive system

To investigate problem $\Sigma_\mu(G_h)$ (Definition 2.1) we shall use a comparison method [1]. For this purpose, we need to construct an integrodifferential equation with fixed moments of impulse actions which is associated with (1). We propose

$$\begin{aligned} dy/dt &= A(t)y + \int_{\alpha}^{t} K(t, s)y(s)ds + C(t)u + f(t) + \mu g(t, y, u, \mu) + \sum_{i:\alpha<\theta_i<t} F_i(t, y, u, \mu), \quad t \neq \theta_i, \\ \Delta y(\theta_i) &= B_i y(\theta_i) + \sum_{j:\alpha<\theta_j<\theta_i} D_{ij}y(\theta_j) + Q_i v_i + J_i + S_i(y, u, v_i, \mu), \end{aligned} \tag{8}$$

where F_i and S_i are some functionals to be determined. Without any loss of generality we may assume that $\tau_i \geq 0$ in G_H.

Definition 4.1 *The systems (1) and (8) are said to have B-property in* G_H *, if for a fixed positive real number* $h < H$ *and a sufficiently small* μ *the following conditions are fulfilled:*

Given any solution $x(t)$ *of (1),* $|x(t)| < h$, $t \in [\alpha, \beta]$, $x(t)$ *meets every surface of discontinuity once, then there is a solution* $y(t)$ *of (8),* $|y(t)| < H$, *such that* $x(t) = y(t)$ *for all* $t \in [\alpha, \beta]$ *except possibly at points* $t \in [\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$. *Conversely, given any solution* $y(t)$ *of (8),* $|y(t)| < h$, $t \in [\alpha, \beta]$, *if a solution* $x(t)$ *of (1),* $x(\alpha) = a, |x(t)| < H$, *intersects every surface of discontinuity once, the condition* $x(t) = y(t)$ *holds for all* $t \in [\alpha, \beta]$ *except possibly at points* $t \in [\theta_i, \zeta_i]$, $i = 1, 2, \ldots, p$.

5 Controllability of impulsive integrodifferential equations

We first consider the controllability problem for (8), (2). This problem is denoted by $\gamma_\mu(G_H)$. Define

$$\Psi(t) = \int_{\alpha}^{t} E(s)E^T(s)ds + \sum_{i:\alpha<\theta_i<t} P_i P_i^T, \tag{9}$$

where $E(t) = \Lambda(\beta, t)C(t)$ and $P_i = \Lambda(\beta, \theta_i+)Q_i$.

Lemma 3 *Let (C1)–(C5) be satisfied. If* $\Psi(\beta)$ *be nonsingular,* $\gamma_\mu(G_H)$ *is a solvable problem.*

We shall need the following condition:

(C6) The inequality

$$\tau_i(y_0(\theta_i),0) > \tau_i(y_0(\theta_i+),0), \tag{10}$$

is valid for all $\{a,b\} \in \overline{G}_h \times \overline{G}_h$, where $\overline{G}_h$ is the closure of G_h, $i = 1,2,\ldots,p$.

Theorem 5 *Let (C1)–(C6) be satisfied. If $\Psi(\beta)$ is nonsingular, then $\Sigma_\mu(G_h)$ is solvable.*

6 Optimal control of response

In this section, we consider the particular case of the equation (1)

$$dx/dt = A(t)x + C(t)u(t) + f(t), \;\; t \neq \theta_i,$$

$$\Delta x(\theta_i) = B_i x(\theta_i) + Q_i v_i + J_i, \quad i = 1,2,\ldots,p, \tag{11}$$

$$x(0) = a, \quad x(\beta) = b, \tag{12}$$

where A and C are defined for any time $t \geq 0$, B_i and Q_i are bounded sequences, and the time $\beta > 0$ is arbitrary. Before than to consider main problem of the section let us to investigate controllability of the problem for a fixed positive number $\beta \in R^n$. We say that the control problem $\mathcal{A}(\beta)$ is solvable if, for any $f, \{J_i\} \in \Pi^m[0,\beta]$ and any $a,b \in R^n$, there exists $\{u,v\} \in \Pi^m[0,\beta]$ for which the boundary value problem (11), (12) has a solution.

Lemma 4 *Let $F, \{V_i\} \in \Pi^m[0,\beta]$. Then the boundary value problem*

$$dx/dt = A(t)x + F(t), \;\; t \neq \theta_i, \quad i = 1,2,\ldots,p,$$

$$\Delta x(\theta_i) = B_i x(\theta_i) + V_i, \tag{13}$$

$$x(0) = 0, \quad x(\beta) = 0, \tag{14}$$

is solvable if and only if, for any solution $y(t)$ of the system

$$dy/dt = -A^T(t)y, \;\; t \neq \theta_i, \quad i = 1,2,\ldots,p,$$

$$\Delta y(\theta_i) = -(I + B_i^T)^{-1} B_i^T y(\theta_i), \tag{15}$$

the following relation holds:

$$\big\langle \{F, V_i\}, \{y, y(\theta_i)\} \big\rangle = 0. \tag{16}$$

Let $Y(t)(y_1, y_2, \ldots, y_n)$, be a fundemental matrix of solutions of adjoint system (15).

Theorem 6 *The control problem $\mathcal{A}(\beta)$ is solvable if and only if*

$$\int_0^\beta Y^T(t)\Big[C(t)u(t)+f(t)\Big]dt + \sum_{i:0<\theta_i<\beta} Y^T(\theta_i)\Big[Q_i v_i + J_i\Big] = Y^T(\beta)b - Y^T(0)a. \qquad (17)$$

Let us fix a positive number β and define the space-product $\Pi^m[0,\beta] = L_2^m[0,\beta] \times D^m[1,p]$ for which θ_i, $i = \overline{1,p}$, are points of discontinuity of functions from $L_2^m[0,\beta]$, which form the ordered sequence in the interval $(0,\beta)$.

We assume that the control $\{u,v\}$ can be chosen only from the set $\Delta \times \Delta' \subset \Pi^m[\alpha,\beta]$ which is bounded in the norm $||\cdot||_{[[0,\beta]}$. The problem of fast response is by using a given element $\{f,J\}$ belonging to the space $\Pi^m[\alpha,\beta]$ for any $\beta > 0$, to find the control $\{u,v\}$ that solves the problem in minimal time:

$$\text{(OCR)}\begin{cases} \text{minimize } \beta \\ \text{subject to (11) and (12).} \end{cases} \qquad (18)$$

We say that a control $\{u,v\}$ with a vector $c = c_0$ in the domain $\Delta \times \Delta'$ satisfies the Pontryagin condition [16] if this control provides in the domain the maximum of the expression $c_0^T Y^T(t)C(t)u(t)$ for almost all $t \in [0,\beta]$, and the maximum of the expression $c_0^T Y^T(\theta_i)Q_i v_i, i = \overline{1,p}$.

Theorem 7 *Let the control $\{u,v\}$ solves the problem of control (11), (12) for the time $\beta > 0$, let it satisfy the Pontryagin condition for some vector $c = c_0$ in the domain $\Delta \times \Delta'$. Suppose that the expression $c_0^T Y^T(t)[C(t)u(t) + f(t) + A(t)]b$ is positive for almost all $t \in [0,\beta]$ and the numbers $c_0^T Y^T(\theta_i)[Q_i v_i + J_i + B_i(I+B_i)^{-1})b], i = \overline{1,p}$, are positive. Then, the control $\{u,v\}$ and trajectory $x(t)$ corresponding to it are optimal for (OCR), i.e., in the sense of fast response.*

Theorem 8 *Suppose that the conditions of the Theorem 7 are satisfied, and $x(t)$ is the optimal trajectory connecting points $x(0) = a$ and $x(\beta) = b$. Then, any part of this trajectory which connects points $x(t_1)$ and $x(t_2), 0 \le t_1 \le t_2 \le \beta$, is also an optimal trajectory.*

References

[1] M.U. Akhmetov, N.A. Perestyuk, On comparison method for differential systems with pulse effect, Differential Equations 9 (26) (1990) 1079–1086.

[2] M.U. Akhmetov, N.A. Perestyuk, Periodic and almost periodic solutions of integro-differential equations with impulse actions, Mat.-Fiz. I Nelin. Mekh. 42 (1987). 5-9.

[3] M.U. Akhmetov, N.A. Perestyuk and M.A. Tleubergenova, The control of linear impulsive systems, Ukrain. Math. Zh. 47 (1995) 307-314.

[4] M. Akhmetov, R. Sejilova, The control of the Boundary Value Problem for Linear Impulsive Integro Differential Systems, Journal of Mathematical Analysis and Applications 236 (1999) 312-326.

[5] M.U. Akhmetov, A. Zafer, The controllability of boundary-value problems for quasilinear impulsive systems, Nonlinear Analysis 34 (1998) 1055–1065.

[6] L. Anokhin, L. Berezansky, E. Braverman, Stability of linear delay impulsive differential equations, Dynamic Systems and Applications 4 (1995) 173–188.

[7] A.A. Aslanyan, Conditions for optimum in problems of control of impulsive systems, Dokl. Acad. Nauk Ukrain. SSR, Ser. A 11 (1982) 3-6.

[8] K. Balachandran, D. Lalitha, Controllability of nonlinear Volterra integro-differential systems with prescribed controls, J. Appl. Math. Stochatic Analysis 5 (1992) 139-146.

[9] A. Bensoussan, J.L. Lions, Nouvelle formulation de problèmes de contr$\hat{o}$le impulsionnel et applications, C.R. Acad Sc. Paris, 276 (1973) 1189-1182.

[10] C. Corduneanu, Sur une equation integrale de la théorie du réglage automatique, C.R. Acad. Sc, Paris, 256 (1967) 3564-3567.

[11] A. Halanay, D. Wexler, Qualitative Theory of Impulse Systems, Editura Academiei Republici Socialiste Romania, Bucuresti, 1968.

[12] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.

[13] N. N. Krasovsky, Theory of Motion Control. Linear Systems, Nauka, Moscow, 1973.

[14] V. Lakshmikantham, D.D. Bainov, P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, New York, 1989.

[15] E. B. Lee, L. Markus, Foundations of Optimal Control Theory, Wiley, New York, 1967.

[16] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience Publishers, John Wiley, New York, 1962.

[17] A. M. Samoilenko, N. A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.

[18] S. Sivasundram, Controllability and Observability of nonlinear integro-differential systems, Proceedings of Dynamic Systems and Applications 2 (1996) 507-516.

[19] G.-W. Weber, Generalized Semi-Infinite Optimization and Related Topics, Heldermann Verlag, Lemgo, 2003.

[20] G.-W. Weber, Generalized Semi-Infinite Optimization: Theory and Applications in Optimal Control and Discrete Optimization, Journal of Statistics & Management Systems 5 (1-3) (2002) 359-388.

[21] G.-W. Weber, New projects at the Institute of Applied Mathematics in METU, "Workshop on advances in continuous optimization", Istanbul, July 4-5, 2003.