Duality in Optimization: From Convex to Nonconvex Problems

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Lagrangian Duality (1)

Primal (P):

$\min f(x)$	f, h_i lower semicontinuous,
s.t. $h_i(x) \leq 0,$	$(i = 1, \ldots, m)$
$x \in X.$	X convex, compact.

Equivalent:

$$\min_{x \in X} \sup_{\lambda \in \mathbb{R}^m_+} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) \right\}$$

Lagrangian

Lagrangian Duality (2)

Exchange min and sup:

Dual problem (D): $$\sup_{\lambda \in \mathbb{R}_+^m} \min_{x \in X} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) \right\}$$

$\lambda_1, \ldots, \lambda_m$ are the dual variables.

The dual objective

$$\Theta(\lambda_1, \ldots, \lambda_m) = \min_{x \in X} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) \right\}$$

is always a concave function.

Is $\sup_{\lambda\in\mathbb{R}^n}\Theta(\lambda)$ attained?

Theorem:

Assume
- f and all h_i l.s.c.
- X compact
- Slater's condition fulfilled for (P).

Then
$$\sup(D) = \max(D).$$

$\mathcal{S} :=$ set of Slater points, $\quad \sup_{\lambda\in\mathbb{R}^n}\Theta(\lambda) = \theta(\bar{\lambda})$

Theorem:

We have
$$\|\bar{\lambda}\|_1 \leq \inf_{\hat{x}\in\mathcal{S}} \frac{\min_{x\in X} f(x) - f(\hat{x})}{\max_i h_i(\hat{x})}.$$

Duality Results

- Linear Program — **Strong Duality:** $\min(P) = \sup(D)$
- Convex Program (+ constraint qualification) — **Strong Duality:** $\min(P) = \sup(D)$
- Nonconvex Program — **Weak Duality:** $\min(P) \geq \sup(D)$

Duality Gap: $\Delta := \min(P) - \sup(D) \geq 0.$

Estimates of the Duality Gap (1)

Aubin/Ekeland (1976):

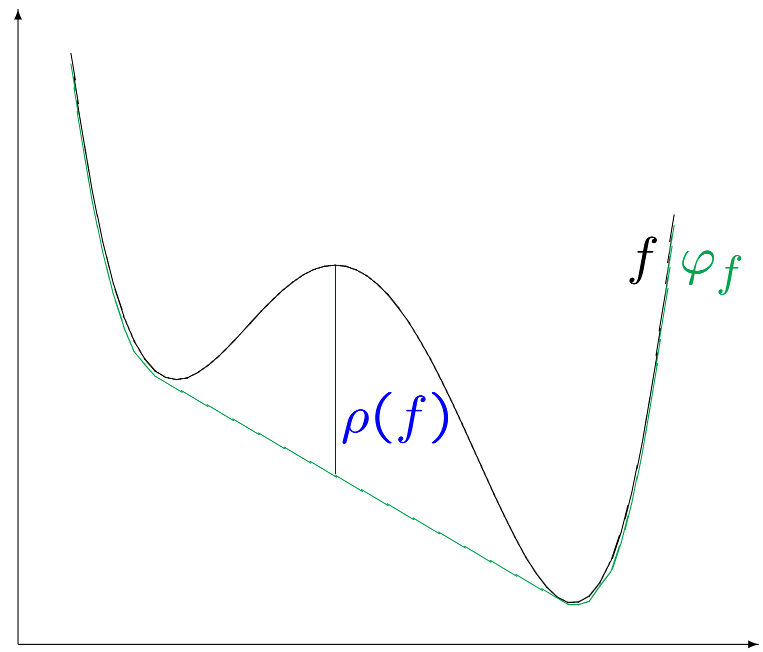
Under certain conditions

$$\Delta \le \rho(f),$$

where

$$\rho(f) := \sup \left\{ f\left(\sum \alpha_i x_i\right) - \sum \alpha_i f(x_i) \right\}.$$

f convex $\iff$ $\rho(f) = 0$.



φ_f: convex envelope of f.

Estimates of the Duality Gap (2)

Horst (1979):

Consider $(\overline{P})$, where f and h_i are substituted by their convex envelopes.

$$\Delta \leq \min(P) - \min(\overline{P}).$$

Consequence:

$$\sup(D) \geq \min(\overline{P}).$$

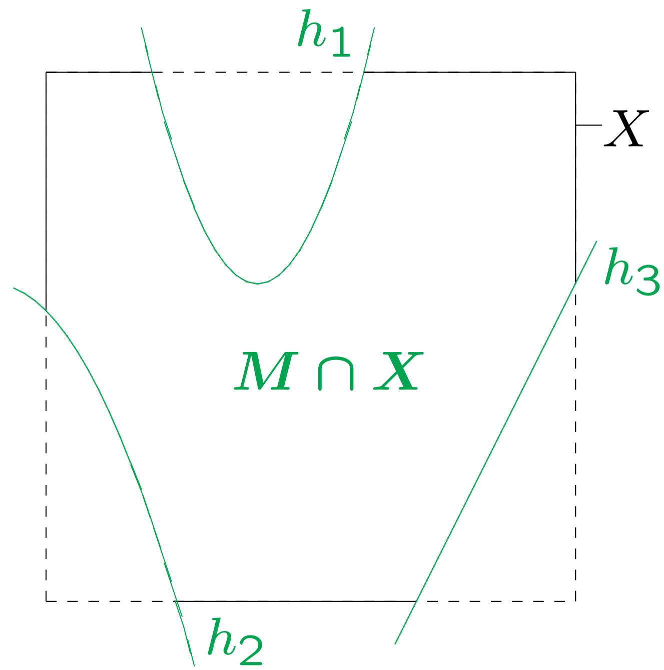
Estimates of the Duality Gap (3)

D. (2001):

$$\Delta \leq \min_{x \in M \cap X} f(x) - \min_{x \in X} f(x),$$

where

$$M := \{x \in \mathbb{R}^n : h_i(x) \leq 0 \ \forall i\}$$



Constraints redundant
$\Rightarrow \Delta = 0.$

Sequences of Duality Gaps

Sequence of primals P_k:

$$\begin{aligned} &\min f(x) \\ \text{s.t. } &h_i(x) \leq 0, \quad i = 1, \ldots, m, \\ &x \in X_k. \end{aligned}$$

f, h_i l.s.c.,
X_k convex, compact,

$X_{k+1} \subseteq X_k$ for all k,

$\emptyset \neq \bigcap_{k \in \mathbb{N}} X_k \subseteq M$.

Sequence of duals D_k: $$\sup_{\lambda \in \mathbb{R}_+^m} \min_{x \in X_k} \left\{ f(x) + \sum_{i=1}^m \lambda_i h_i(x) \right\}.$$

Then:

$$\lim_{k \to \infty} \Delta_k = 0.$$

Primal Infeasibility (1)

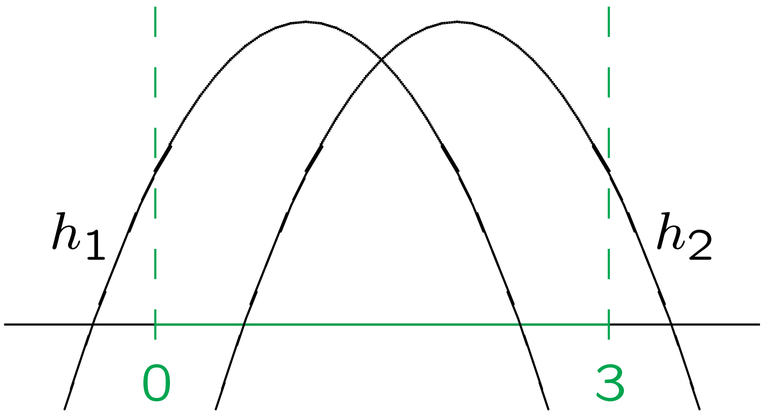
• Linear Program	(P) infeasible	$\Rightarrow$	$\sup(D) = \pm\infty$.
• Convex Program (X compact)	(P) infeasible	$\Rightarrow$	$\sup(D) = \pm\infty$
• Nonconvex Program	(P) infeasible	$\Rightarrow$	**??**

Primal Infeasibility (2)

Counterexample:

$$\begin{aligned} & \min x \\ \text{s.t.} \quad & h_1(x) = -x^2 + 2x + 1 \leq 0, \\ & h_2(x) = -x^2 + 4x - 2 \leq 0, \\ & x \in [0, 3] \end{aligned}$$

is infeasible.



Dual: $$\sup_{\lambda \in \mathbb{R}^2_+} \min_{x \in [0,3]} \left\{ (-\lambda_1 - \lambda_2) x^2 + (1 + 2\lambda_1 + 4\lambda_2) x + (\lambda_1 - 2\lambda_2) \right\}$$

But: $\sup(D) = 1$, attained at $\lambda_1 = 1, \lambda_2 = 0$ and $x = 0$.

Therefore:

$$(P) \text{ infeasible } \not\Rightarrow \sup(D) = \pm\infty.$$

Primal Infeasibility (3)

However . . .

Example 1:

$$\begin{aligned} &\min x \\ \text{s.t.}\quad & -x^2 + 2x + 1 \le 0, \\ & -x^2 + 4x - 2 \le 0, \\ & x \in [0, 3] \end{aligned}$$

infeasible, $\sup(D) = 1$.

Example 2:

$$\begin{aligned} &\min x \\ \text{s.t.}\quad & -x^2 + 2x + 1 \le 0, \\ & -x^2 + 4x - 2 \le 0, \\ & x \in [0, 3/2] \end{aligned}$$

infeasible, $\sup(D) = +\infty$.

Why ???

Primal Infeasibility (4)

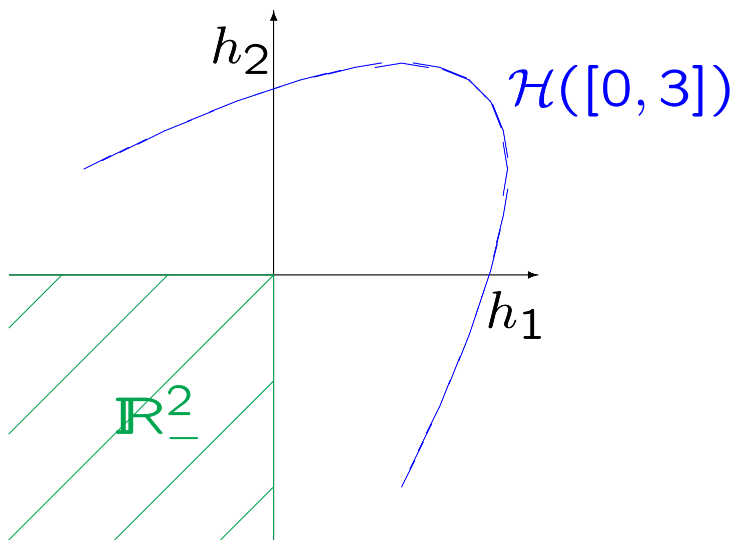
Image space analysis:

$$\mathcal{H}(X) = \{y \in \mathbb{R}^m : y_i = h_i(x) \text{ for some } x \in X, i = 1, \ldots, m\}.$$

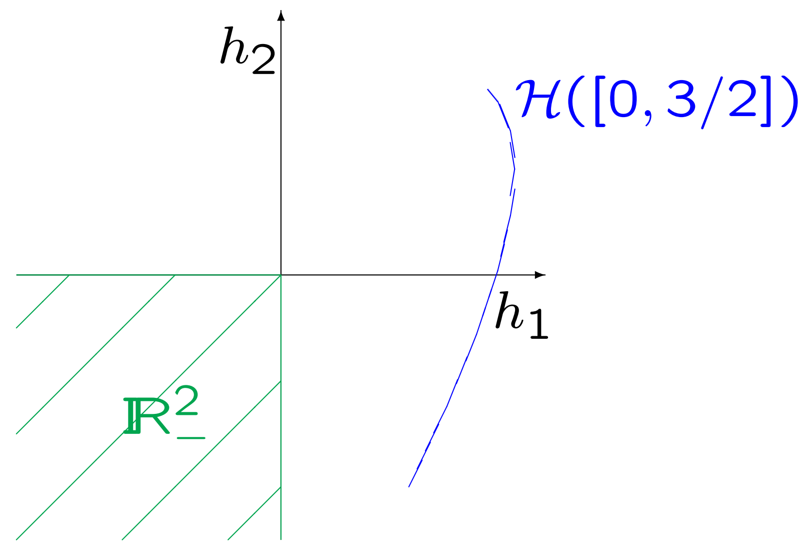
Clearly:

$$(P) \text{ infeasible } \iff \mathcal{H}(X) \cap \mathbb{R}^m_- = \emptyset.$$

For Example 1:



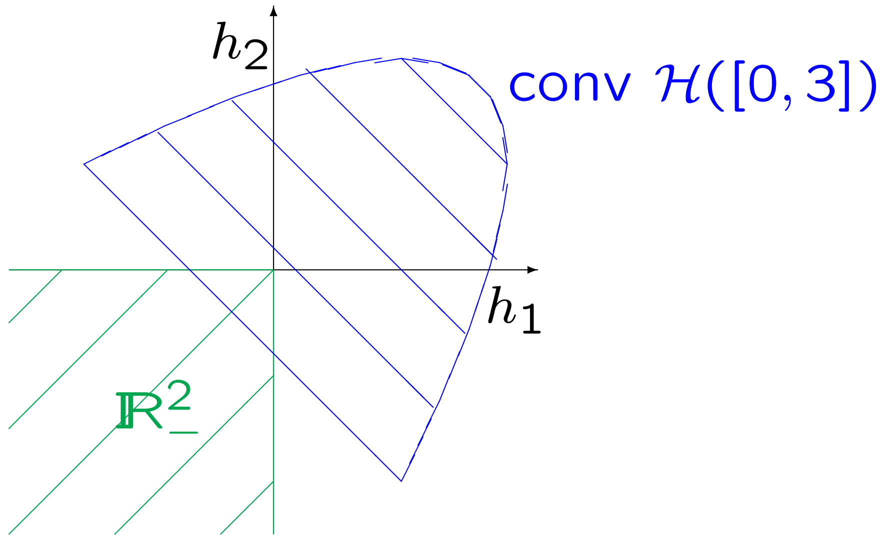
For Example 2:



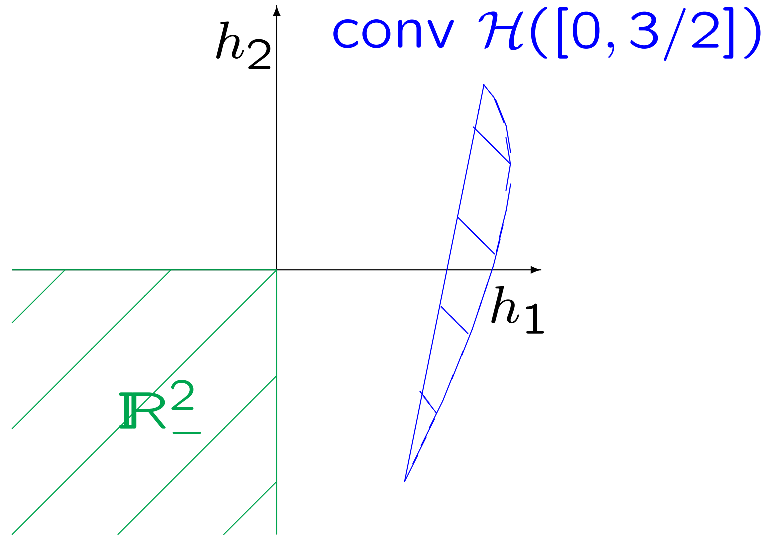
Primal Infeasibility (5)

Now take convex hulls:

For Example 1:



For Example 2:



Theorem: If f, h_i continuous and X compact, then

$$\sup(D) = +\infty \iff \text{conv}(\mathcal{H}(X)) \cap \mathbb{R}^m_- = \emptyset.$$

Dual versus Underestimation Bounds (1)

Always: $\text{sup}(D) \geq \text{min}(\overline{P})$.

Falk (1969):
If constraints h_i linear and strong duality holds for $(\overline{P})$, then

$$\text{sup}(D) = \text{min}(\overline{P}).$$

When is $\text{sup}(D) > \text{min}(\overline{P})$ **??**

Dual versus Underestimation Bounds (2)

$$(P) \qquad \begin{array}{c} \min -x^2 \\ x^2 - x - 2 \le 0, \;\; x \in [-2,3] \end{array}$$

$\min(P) = -4$, attained at $x = 2$.

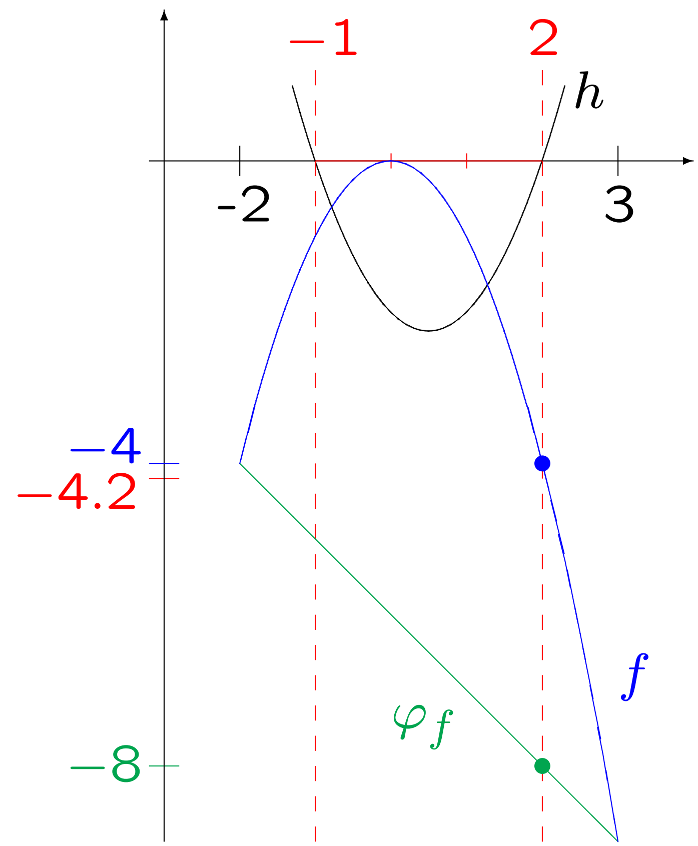
$$(\bar{P}) \qquad \begin{array}{c} \min -x - 6 \\ x^2 - x - 2 \le 0, \;\; x \in [-2,3] \end{array}$$

$\min(\bar{P}) = -8$, attained at $x = 2$.

Dual (D):

$$\sup_{\lambda \in \mathbb{R}_+} \min_{x \in [-2,3]} \{(\lambda - 1)x^2 - \lambda x - 2\lambda\},$$

$\sup(D) = -4.2$ at $\lambda = 6/5$ and $x = 3$



Dual versus Underestimation Bounds (3)

Theorem:
Let $f : \mathbb{R}^n \to \mathbb{R}$ be strictly concave, $h_i(x) : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, m)$ be strictly convex and continuously differentiable.
Let $X \subset \mathbb{R}^n$ be convex and compact.
Let Slater's condition be fulfilled for (P), and $\min(P) > \min(\bar{P})$.
Assume that the convex envelope φ_f of f on X is not constant on any interval contained in X.

Then the dual bound is strictly better than the convex envelope bound, i.e.

$$\sup(D) > \min(\overline{P}).$$

How to solve (D)?

The dual objective

$$\Theta(\lambda_1, \ldots, \lambda_m) = \min_{x \in X} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) \right\}$$

is always a concave function.

(D) is a convex problem

can be solved with bundle methods.

In some cases: (D) transforms to an LP:

- Generalized Bilinear Constraints
- Concave Minimization under Reverse Convex Constraints
- Maximizing the Sum of Affine Ratios

Application to Branch–and–Bound (1)

Step 0. Compute a compact set $X \supset M$ of simple structure. Compute a lower bound $\ell(X) \leq \min_{x \in X \cap M} f(x)$ for $\min(P)$, as well as an upper bound $u(X) \geq \min_{x \in X \cap M} f(x)$. Let $\mathcal{L}$ be an empty list, and set $k := 1$.

Step 1. Partition X into ν compact subsets $X_1, \ldots, X_\nu$, and add them to the list $\mathcal{L}$.

Step 2. Calculate lower bounds $\ell(X_i) \leq \min_{x \in X_i \cap M} f(x)$ and upper bounds $u(X_i) \geq \min_{x \in X_i \cap M} f(x)$ for all newly generated sets X_i.

Application to Branch–and–Bound (2)

Step 3. Update the current best lower and upper bounds: Put

$$\ell_k := \min_{X \in \mathcal{L}} \ell(X) \qquad \text{and} \qquad u_k := \min_{X \in \mathcal{L}} u(X).$$

Step 4. Discard elements from $\mathcal{L}$ which can not contain a global optimizer, i.e. discard all elements X with the property (i) $X \cap M = \emptyset$, or (ii) $\ell(X) > u_k$.

Step 5. Select a new $X \in \mathcal{L}$ which is to be subdivided in the next iteration, and remove it from $\mathcal{L}$.

Step 6. While stopping criteria are not fulfilled, increment $k := k + 1$, and go to Step 1.

Application to Branch–and–Bound (3)

Theorem:
Let M be nonempty and compact, let $X_1 \supset M$. Let $f : X_1 \to \mathbb{R}$ be l.s.c. and let $\{X_k\}_{k \in \mathbb{N}}$ be a nested sequence of nonempty, compact sets converging to $\emptyset \neq X_\infty \subset M$. Then

$$\lim_{k \to \infty} [\sup(D_k)] = \min_{x \in X_\infty} f(x).$$

It follows that

Branch–and–Bound algorithms using dual bounds
are convergent.