Solving the least squares method problem in the AHP

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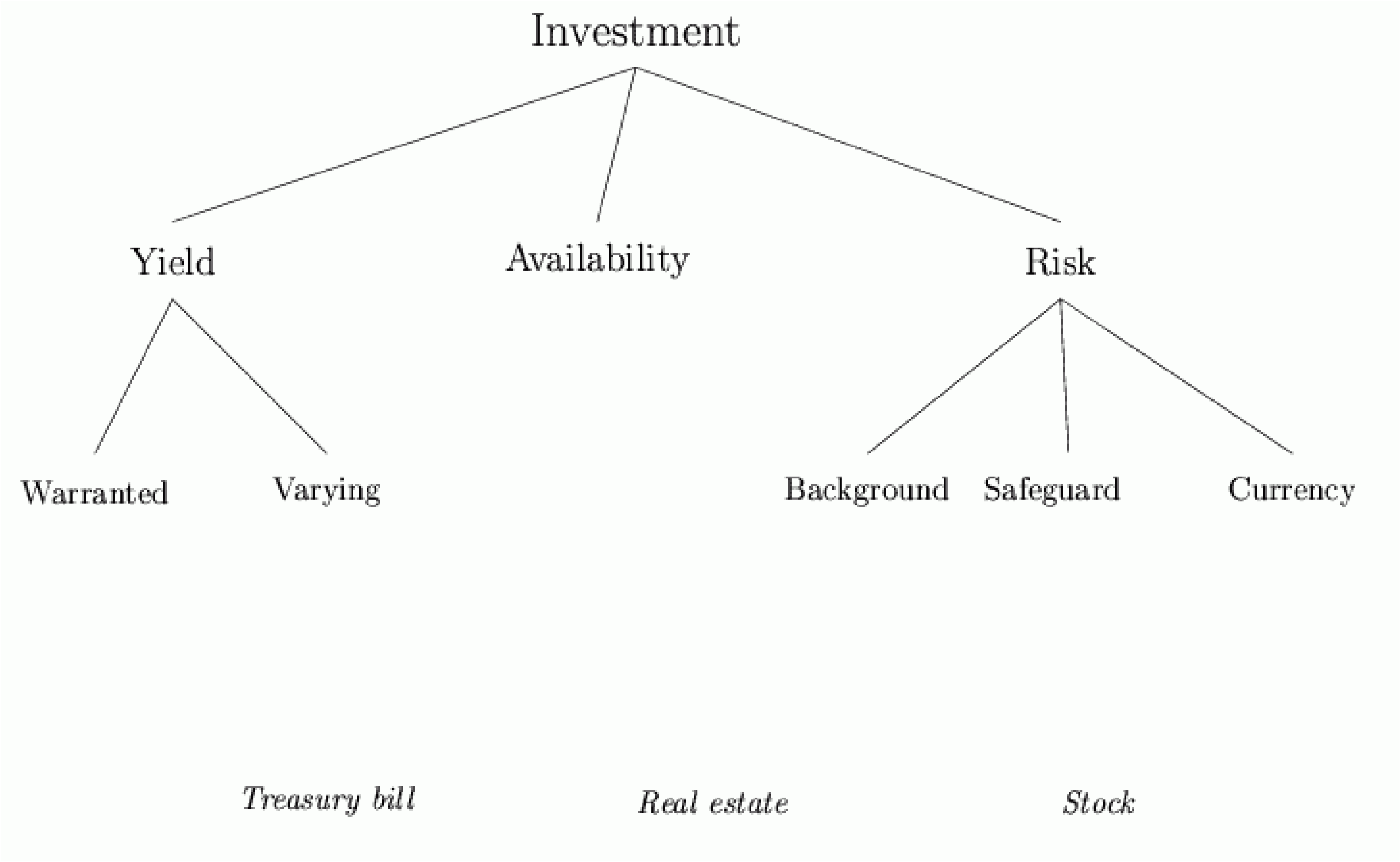
Multicriteria Decision Models

Analytic Hierarchy Process

Criterion tree

Pairwise comparison method


Investment
Yield
Availability
Risk
Warranted
Varying
Background
Safeguard
Currency
Treasury bill
Real estate
Stock

Pairwise comparison matrix:

Given $w_1, w_2, w_3, \dots, w_n$ weights,

$$\begin{pmatrix} 1 & \frac{w_1}{w_2} & \frac{w_1}{w_3} & \dots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & 1 & \frac{w_2}{w_3} & \dots & \frac{w_2}{w_n} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & 1 & \dots & \frac{w_3}{w_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \dots & 1 \end{pmatrix},$$

where for any i, j, k indices

$$w_{ij} > 0, \quad w_{ji} = \frac{1}{w_{ij}}, \quad w_{ij} = w_{ik} w_{kj}.$$

In practical problems

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 1 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix},$$

is given, where for any $i, j = 1, \dots, n$ indices

$$a_{ij} > 0, \qquad a_{ij} = \frac{1}{a_{ji}}.$$

The aim is to find the $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n_+$ weight vector.

Methods for determining weights:

Eigenvector Method (Saaty): $Aw = \lambda_{max} w$.

Least Squares Method, (LSM):

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} - \frac{w_i}{w_j} \right)^2$$

$$\sum_{i=1}^{n} w_i = 1,$$

$$w_i > 0, \qquad i = 1, 2, \ldots, n.$$

Methods for determining weights:

Weighted Least Squares Method (WLSM)

Logarithmic Least Squares Method (LLSM)

Goal Programming Method

Fuzzy Programming Method

Singular Value Decompositon

LSM problem:

Given $\mathbf{A}$ $n \times n$ pairwise comparison matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 1/a_{12} & 1 & a_{23} & \dots & a_{2n} \\ 1/a_{13} & 1/a_{23} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/a_{1n} & 1/a_{2n} & 1/a_{3n} & \dots & 1 \end{pmatrix}.$$

We look for $\mathbf{X}$ consistent matrix, which is the closest one to $\mathbf{A}$ in Frobenius-norm.

$$\mathbf{X} = \begin{pmatrix} 1 & w_1/w_2 & w_1/w_3 & \dots & w_1/w_n \\ w_2/w_1 & 1 & w_2/w_3 & \dots & w_2/w_n \\ w_3/w_1 & w_3/w_2 & 1 & \dots & w_3/w_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & w_n/w_3 & \dots & 1 \end{pmatrix}.$$

$$\min \| \mathbf{A} - \mathbf{X} \|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij} - \frac{w_i}{w_j} \right)^2,$$

where

$$\begin{aligned} w_1 + w_2 + \cdots + w_n &= 1, \\ w_1, w_2, \dots, w_n &> 0. \end{aligned}$$

Objective function:

Nonlinear, non-convex, with multiple minima.

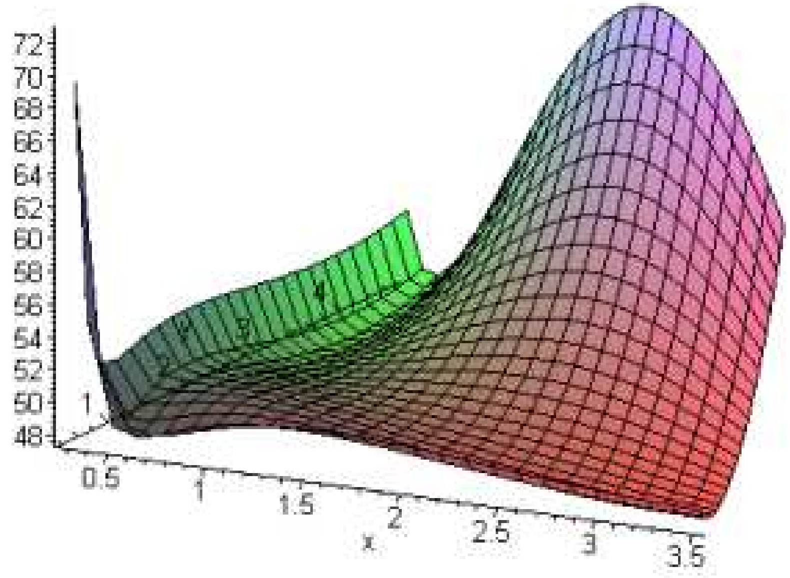
Jensen (1984) mentions the difficulty of problem.

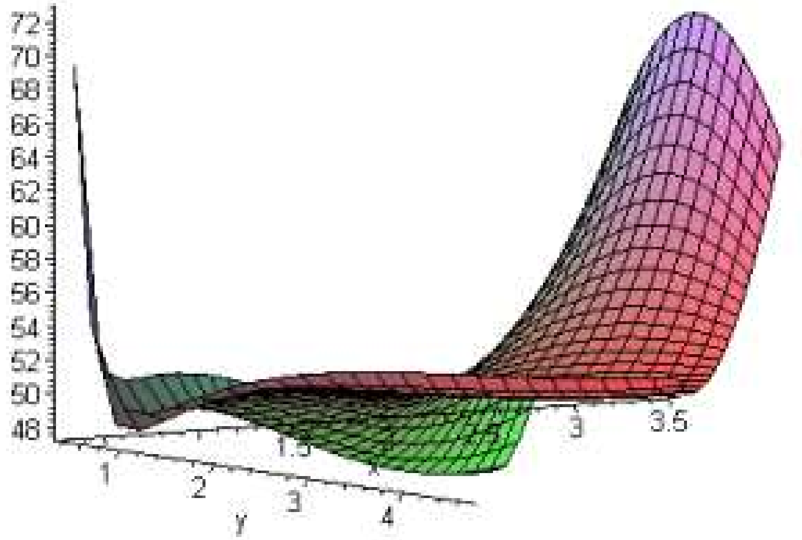
András Farkas (2001) finds optimum by Newton-approximation. A good initial point is needed.

Bozóki (2003) solves the 3×3 case.

Bozóki, Robert Lewis (2004): the 4×4-es case.

LSM objective function





With new variables $x_1, x_2, \dots, x_{n-1}$,

$$x_1 = \frac{w_1}{w_2}, \quad x_2 = \frac{w_1}{w_3}, \quad \dots, \quad x_i = \frac{w_1}{w_{i+1}}, \quad \dots, \quad x_{n-1} = \frac{w_1}{w_n}$$

the number of variables is reduced by 1:

$$\begin{aligned} \min \quad & f(x_1, x_2, \dots, x_{n-1}) \\ & x_1, x_2, \dots, x_{n-1} > 0. \end{aligned}$$

The first-order conditions of optimality:

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_{n-1}} = 0.$$

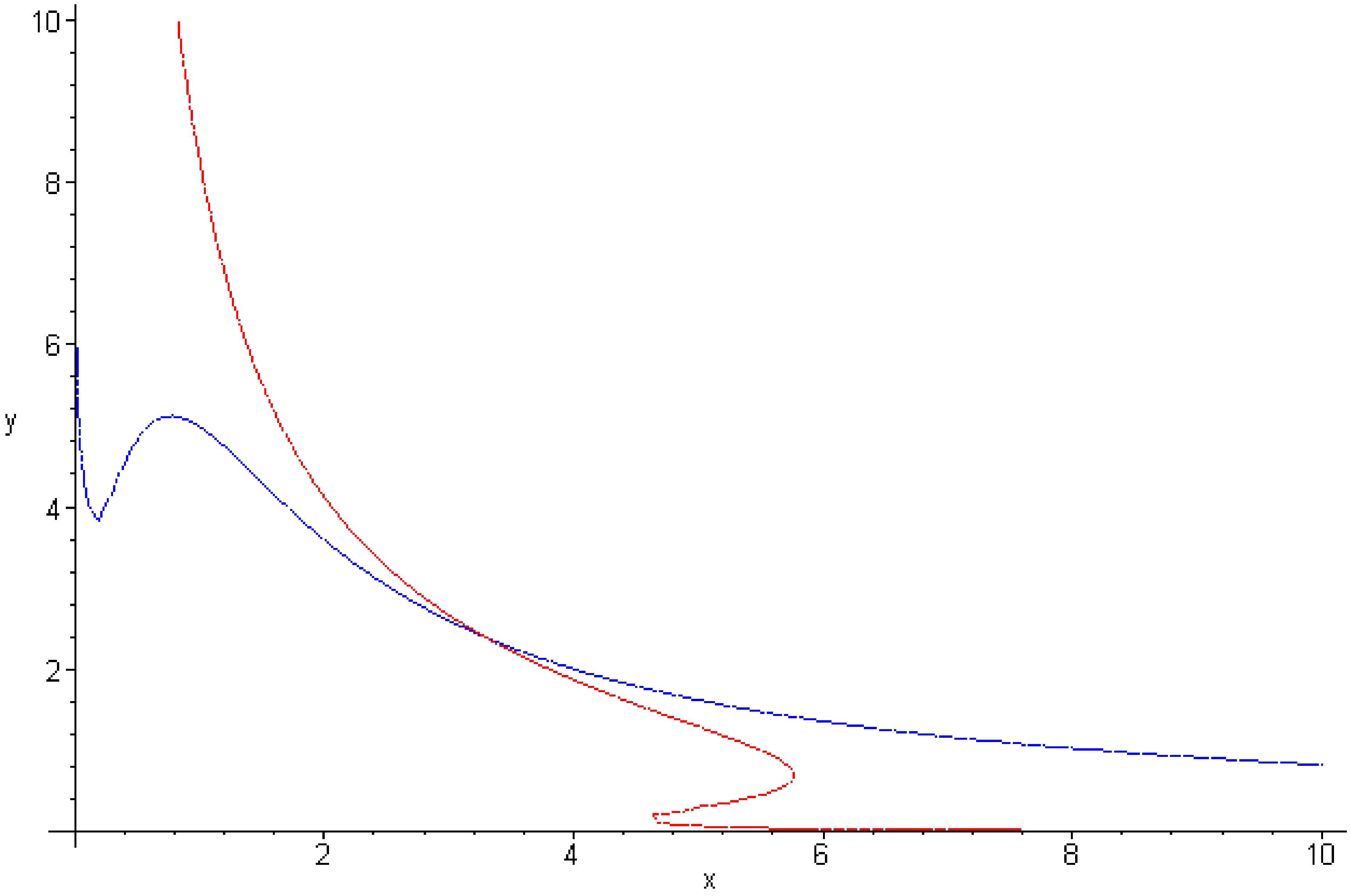
The first partial derivatives of f are rational functions of variables $x_1, x_2, \ldots, x_{n-1}$.

Multiplying by the common denominator we have the polynomials $P_1, P_2, \ldots, P_{n-1}$:

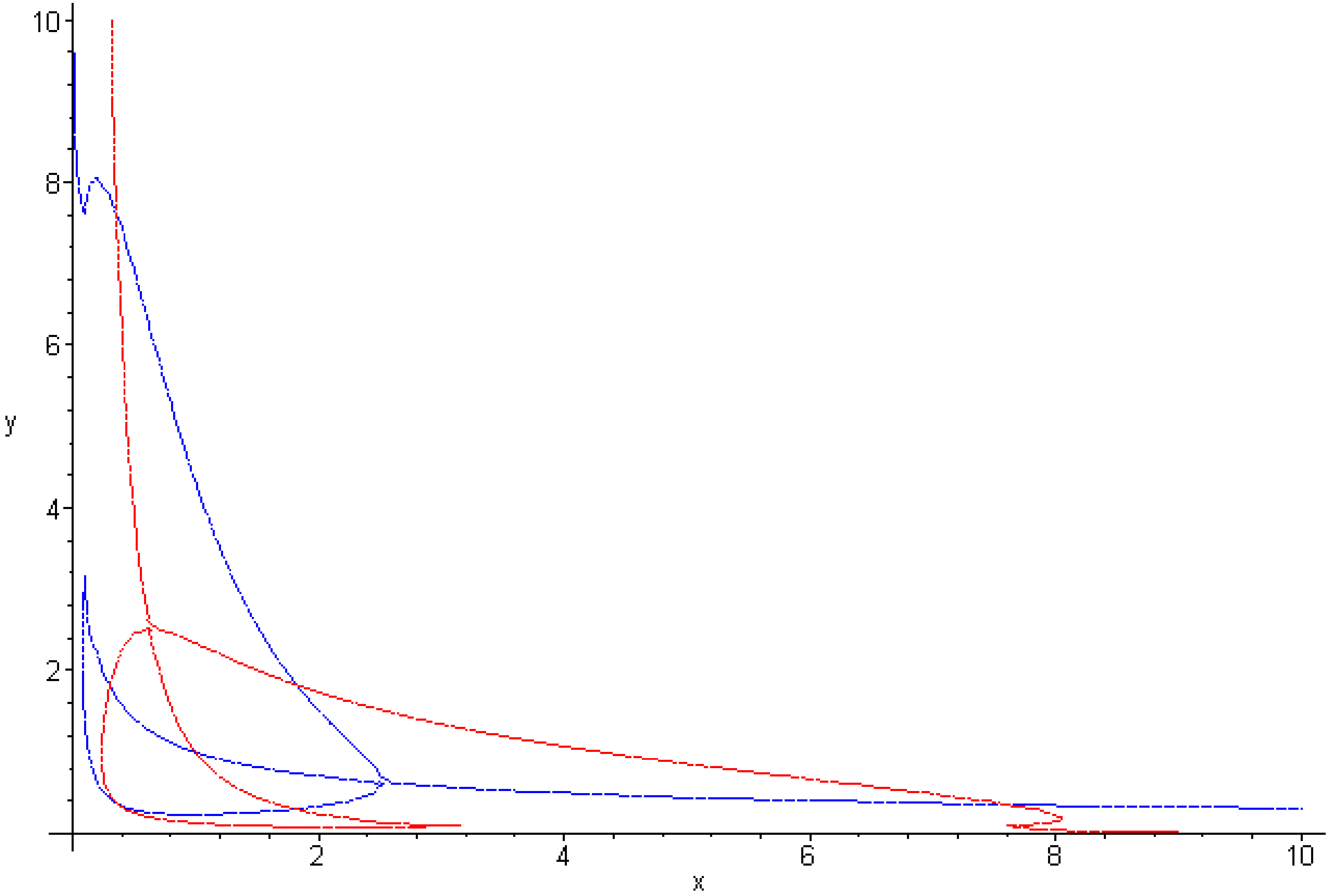
$$P_i(x_1, x_2, \ldots, x_{n-1}) = \frac{1}{2} x_i \frac{\partial f}{\partial x_i} \prod_{j=1}^{n} {x_j}^2, \qquad i = 1, 2, \ldots, n-1.$$

We are interested only in $(x_1, x_2, \ldots, x_{n-1}) > 0$ solutions, so an equivalent system can be written as:

$$
\begin{aligned}
P_1(x_1, x_2, \ldots, x_{n-1}) &= 0 \\
P_2(x_1, x_2, \ldots, x_{n-1}) &= 0 \\
&\vdots \\
P_{n-1}(x_1, x_2, \ldots, x_{n-1}) &= 0
\end{aligned}
$$



$$P_1(x,y) = 0, \qquad P_2(x,y) = 0.$$



$$P_1(x,y) = 0, \qquad\qquad P_2(x,y) = 0.$$

Size of matrix ($n \times n$)	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
Number of terms in the objective function f ($n^2 - n$)	6	12	20	30	42	56	72	90
Number of terms in P_i $i = 1, 2, \ldots, n-1$	8	12	16	20	24	28	32	36
Degree(P_i, x_j) $i = 1, 2, \ldots, n-1$ $j = 1, 2, \ldots, n-1$	4	4	4	4	4	4	4	4
Total degree of P_i $i = 1, 2, \ldots, n-1$	6	8	10	12	14	16	18	20
Minimal total degree in terms of P_i $i = 1, 2, \ldots, n-1$	2	4	6	8	10	12	14	16

Table 1. Properties of polynomial systems, $n = 3, 4, \ldots, 10$

Solution of polynomial systems

	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
Resultant method (Bozóki)	✓	-	-	-	-	-	-
Gröbner basis	✓	-	-	-	-	-	-
Generalized resultants (Bozóki, Robert Lewis)	✓	✓	-	-	-	-	-
Homotopy method (Tien-Yien Li, Tangan Gao)	✓	✓	✓	✓	✓	✓	-

	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
CPU time	0.05 sec	0.5 sec	20 sec	14 min	10 hours	3 day
Number of common roots in $\mathbb{C}^n$	24	224	1840	14000	10^5	10^6
Number of common roots in $\mathbb{R}^n_+$	$1-4$	1^*	1^*	1^*	1^*	1^*

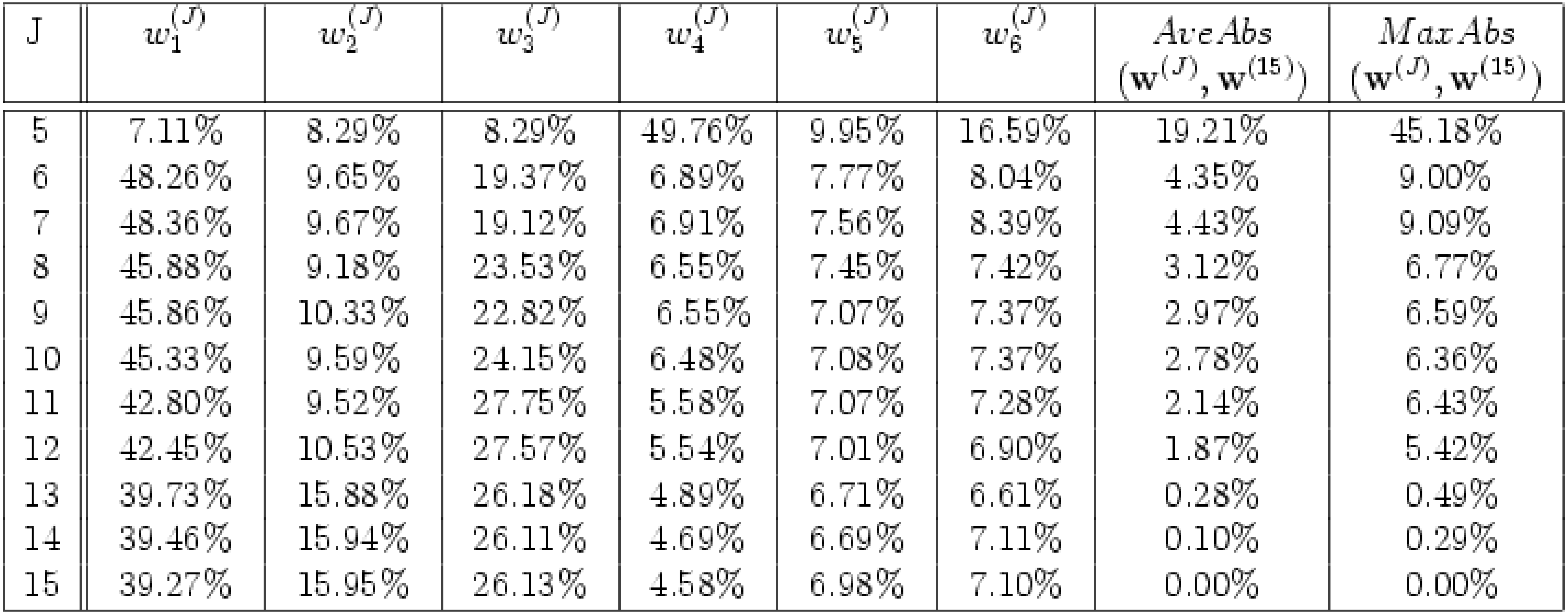
I thank to the Supercomputer Center of National Information Infrastructure Development Program (NIIF).

$$\begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & 1 & \frac{1}{3} & 5 & 3 & 3 & \frac{1}{5} & \frac{1}{7} \\ \frac{1}{3} & 3 & 1 & 6 & 3 & 4 & 6 & \frac{1}{5} \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{6} & 1 & \frac{1}{3} & \frac{1}{4} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & 3 & 1 & \frac{1}{2} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & 4 & 2 & 1 & \frac{1}{5} & \frac{1}{6} \\ 3 & 5 & \frac{1}{6} & 7 & 5 & 5 & 1 & \frac{1}{2} \\ 4 & 7 & 5 & 8 & 6 & 6 & 2 & 1 \end{pmatrix}$$

w^{EM}	w^{LSM}
0.173	0.220
0.054	0.047
0.188	0.149
0.018	0.029
0.031	0.041
0.036	0.042
0.167	0.203
0.333	0.269

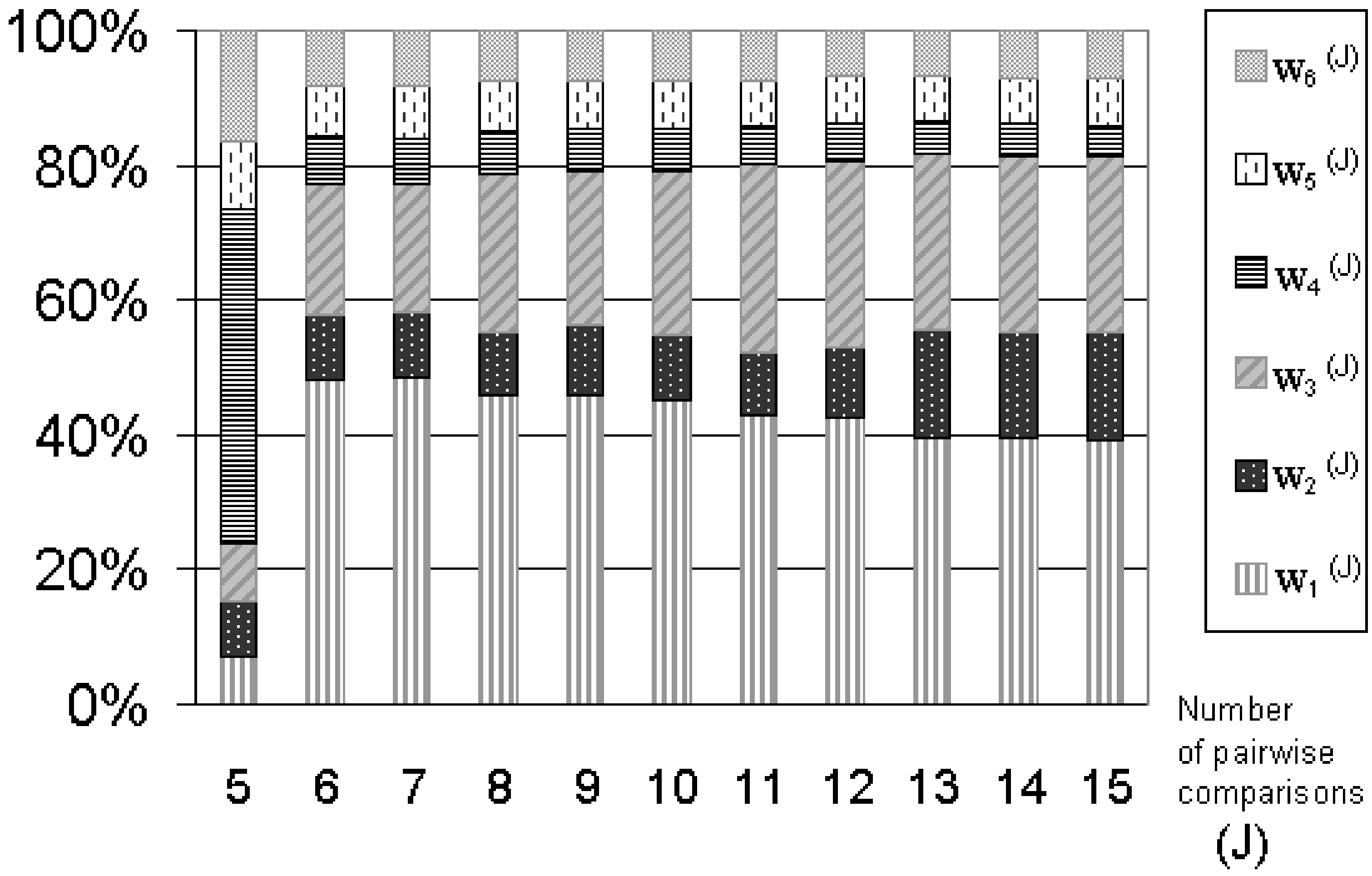
$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 3 & 7 & 6 & 6 \\ \frac{1}{5} & 1 & \frac{1}{3} & 5 & 3 & 3 \\ \frac{1}{3} & 3 & 1 & 6 & 3 & 4 \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{6} & 1 & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & 3 & 1 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & 4 & 2 & 1 \end{pmatrix}$$

$$\begin{array}{c} w^{LSM} \\ 39.27\% \\ 15.95\% \\ 26.13\% \\ 4.58\% \\ 6.98\% \\ 7.10\% \end{array}$$



J	$w_1^{(J)}$	$w_2^{(J)}$	$w_3^{(J)}$	$w_4^{(J)}$	$w_5^{(J)}$	$w_6^{(J)}$	$AveAbs$ $(\mathbf{w}^{(J)}, \mathbf{w}^{(15)})$	$MaxAbs$ $(\mathbf{w}^{(J)}, \mathbf{w}^{(15)})$
5	7.11%	8.29%	8.29%	49.76%	9.95%	16.59%	19.21%	45.18%
6	48.26%	9.65%	19.37%	6.89%	7.77%	8.04%	4.35%	9.00%
7	48.36%	9.67%	19.12%	6.91%	7.56%	8.39%	4.43%	9.09%
8	45.88%	9.18%	23.53%	6.55%	7.45%	7.42%	3.12%	6.77%
9	45.86%	10.33%	22.82%	6.55%	7.07%	7.37%	2.97%	6.59%
10	45.33%	9.59%	24.15%	6.48%	7.08%	7.37%	2.78%	6.36%
11	42.80%	9.52%	27.75%	5.58%	7.07%	7.28%	2.14%	6.43%
12	42.45%	10.53%	27.57%	5.54%	7.01%	6.90%	1.87%	5.42%
13	39.73%	15.88%	26.18%	4.89%	6.71%	6.61%	0.28%	0.49%
14	39.46%	15.94%	26.11%	4.69%	6.69%	7.11%	0.10%	0.29%
15	39.27%	15.95%	26.13%	4.58%	6.98%	7.10%	0.00%	0.00%

$$AveAbs(\mathbf{u},\mathbf{v}) = \frac{1}{6}\sum_{i=1}^{6} |u_i - v_i|; \qquad MaxAbs(\mathbf{u},\mathbf{v}) = \max_{1\le i\le 6} |u_i - v_i|.$$


100%
80%
60%
40%
20%
0%
5
6
7
8
9
10
11
12
13
14
15
Number of pairwise comparisons (J)
$\mathbf{w}_6^{(J)}$
$\mathbf{w}_5^{(J)}$
$\mathbf{w}_4^{(J)}$
$\mathbf{w}_3^{(J)}$
$\mathbf{w}_2^{(J)}$
$\mathbf{w}_1^{(J)}$

Questions:

Analysis of LSM compared to the other weighting methods, particularly the Eigenvector Method.

Identification of types of decision problems.