# Abnormal Extrema in Non Linear Programming Problems: Optimality Conditions and Rigidity

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$J_A(\hat{x}) = \{j \in J : \;\, g_j(\hat{x}) = 0\}$ – index set of active constraints.

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where $(\lambda_0, \lambda, \mu) \in \mathbb{R}^{s+t+1},\ \lambda_0 \geq 0, \mu \geq 0.$

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such that the next conditions are satisfied:

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**If in addition $\hat{x}$ is the regular point of the active constraints, then $\hat{\lambda}_0 \neq 0$*.**

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$$\sum_{i=1}^{s} \hat{\lambda}_i \nabla h_i(\hat{x}) + \sum_{j \in J_A} \hat{\mu}_j \nabla g_j(\hat{x}) = 0,$$

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**Note.**

$\hat{x}$ extremum $\leftrightsquigarrow$ $\Lambda(\hat{x})$ – the set of Lagrange multipliers correspondent to $\hat{x}$.

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If $\hat{x}$ is a local minimizer in (1) and if $\hat{x}$ is a **regular** point of active constraints, then there exists

$$(\hat{\lambda}_0, \hat{\lambda}, \hat{\mu}) \in \mathbb{R}^{s+t+1} \setminus \{0\}, \quad \hat{\lambda}_0 \geq 0,\ \hat{\mu} \geq 0$$

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Let $\hat{x} \in S$.
If there exists $(\hat{\lambda}_0, \hat{\lambda}, \hat{\mu}) \in \mathbb{R}^{s+t+1} \backslash \{0\}$, with $\hat{\lambda}_0 > 0,\ \hat{\mu} \geq 0$ such that:

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3. Agrachev A.A., Sarychev A.V. *Abnormal Subriemanian Geodesics: Morse Index and Rigidity*, Ann.Inst.Henri Poincare–Analyse Non Lineaire, (1996) Vol.13, p.635-690.

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**Theorem 1. (Sarychev A., Tchemisova T.)**

Consider the problem (1) with $f,\ h_i,\ g_j \in C^2(\mathbb{R}^n)$.
Let $\hat{x} \in S$. Suppose that there exists some vector

$$(\hat{\lambda}_0, \hat{\lambda}, \hat{\mu}) \in \mathbb{R}^{s+t+1} \setminus \{0\},\ \hat{\lambda}_0 \geq 0,\ \hat{\mu} \geq 0$$

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# Lagrange problem of Calculus of Variations

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$$\int_0^T \varphi(x(t), u(t))dt \longrightarrow min,$$
$$\dot{x} = f(x, u),$$
$$x(0) = x^0, x(T) = x^1.$$

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We have to prove that $\hat{x} = 0$ is isolated in $S$.

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$\exists \sigma > 0, \exists \varepsilon > 0$ such that $\forall x \in V_\varepsilon(0)$ :

$$\max\{\| H(x) \|, g_1(x), ..., g_t(x)\} \geq \sigma \| x \|^2 . \qquad (2)$$

# Proof of Theorem 1.

**Proposition 1.** *Suppose that there exists $\delta > 0$, $\varepsilon > 0$ such that for each $x \in V_\varepsilon(0)$:*

$$\mathcal{L}(x) + \max\{\| H(x) \|, g_1(x), ..., g_t(x)\} \geq \delta \| x \|^2, \tag{3}$$

*where $\mathcal{L}(x) = \mathcal{L}(x, 0, \hat{\lambda}, \hat{\mu})$.*

*Then the inequality (2) is valid for some $\sigma > 0$.*

Splitting of $\mathbb{R}^n$:

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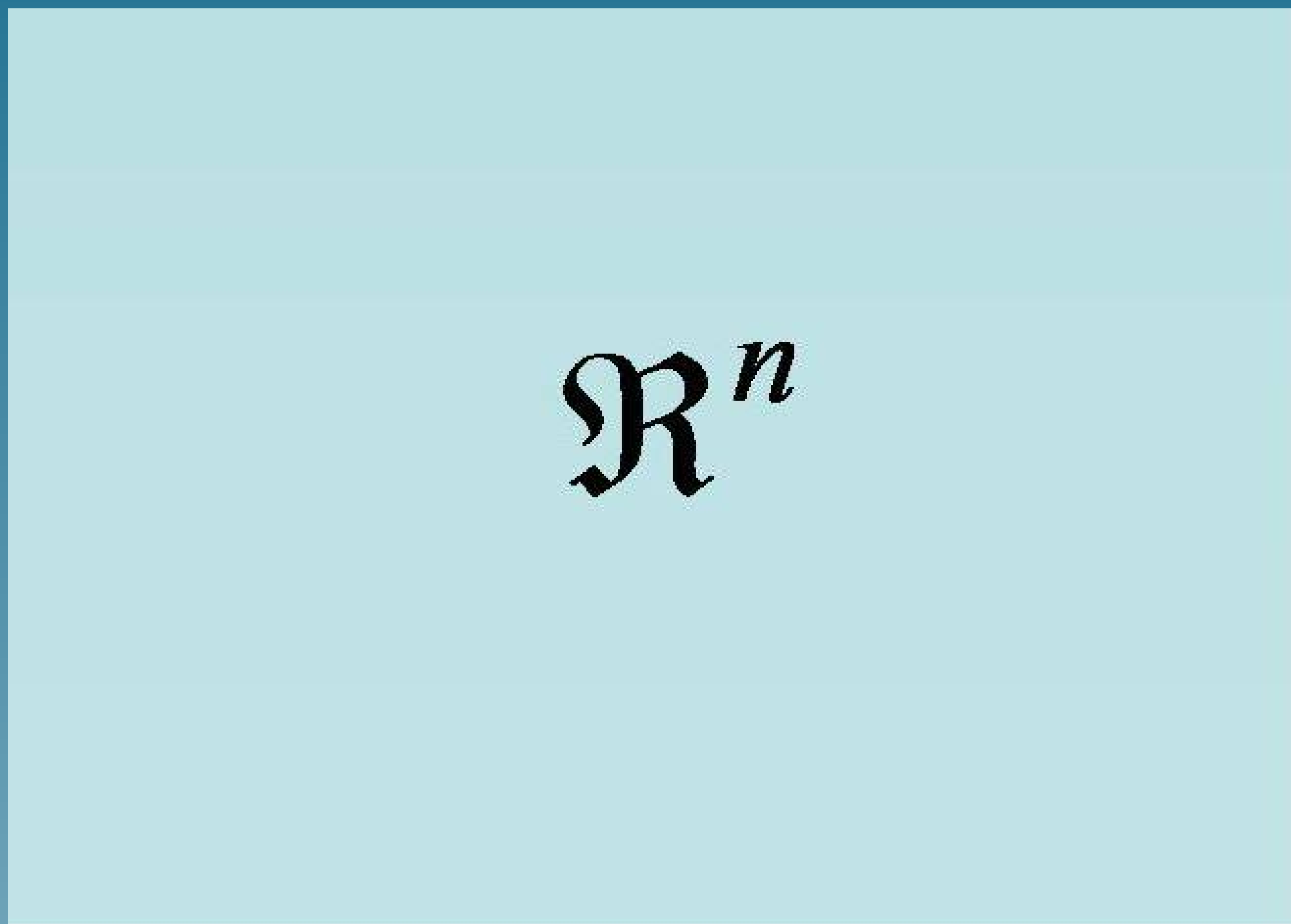

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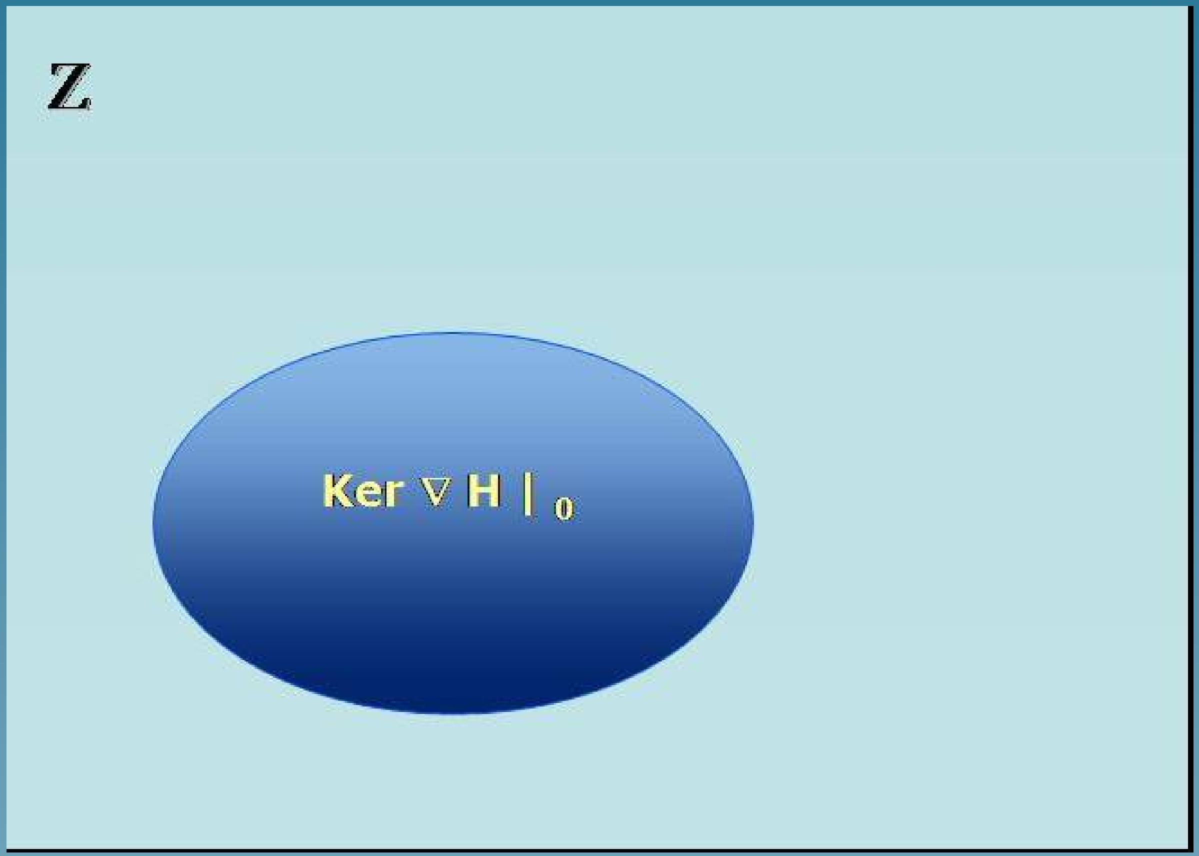

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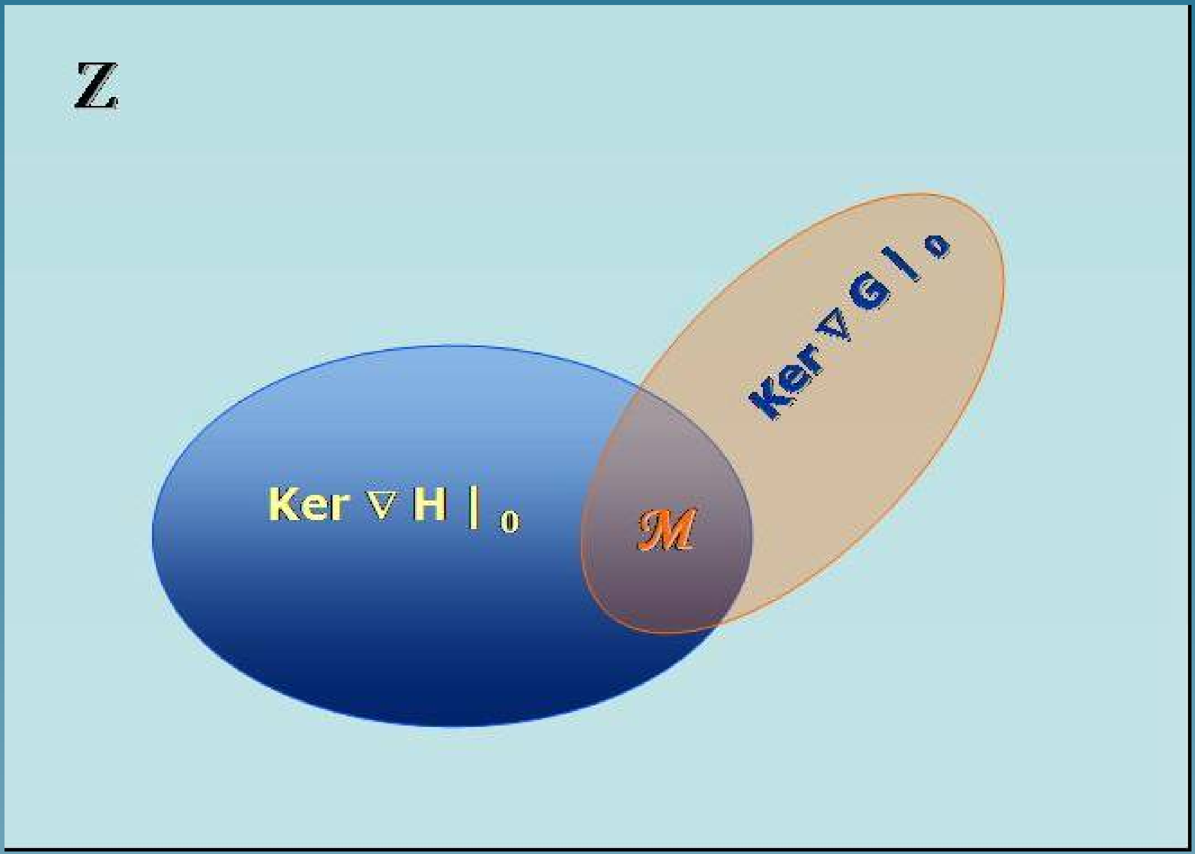

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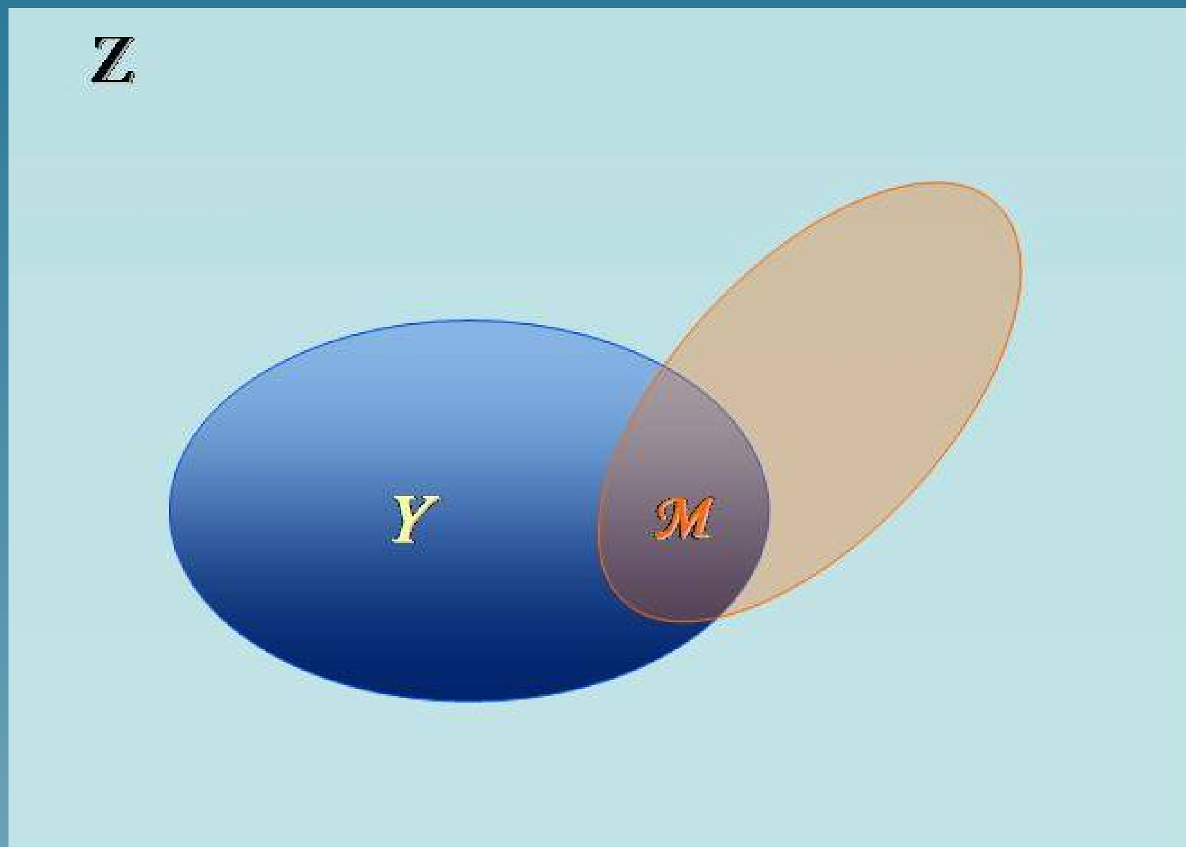

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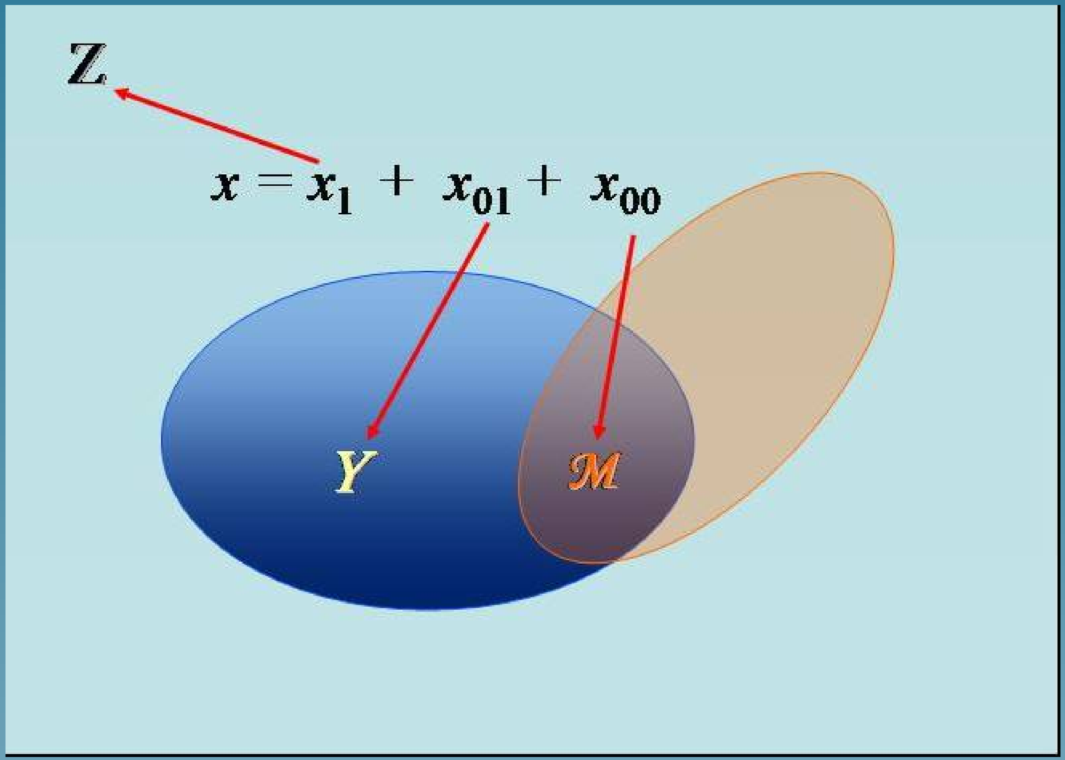

**Lemma 1.** *For some* $a > 0,\ \beta > 0$

$$\mathcal{L}(x) \geq \max(0,\ a \parallel x_{00} \parallel^2 - \beta(\parallel x_{01} \parallel + \parallel x_1 \parallel) \parallel x \parallel). \tag{4}$$

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**Lemma 2.** *For some* $c > 0,\ A > 0$

$$\parallel H(x) \parallel \geq max(0, c \parallel x_1 \parallel - A(\parallel x_{01} \parallel^2 + \parallel x_{00} \parallel^2). \quad (5)$$

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**Lemma 3.** *For some* $b > 0, \nu > 0, \omega > 0$ *there holds:*

$$\max\{g_1(x), g_2(x), ..., g_t(x)\} \geq$$
$$\geq \max\{0, b \parallel x_{01} \parallel - \nu \parallel x_1 \parallel - \omega \parallel x_{00} \parallel^2\}. \qquad (6)$$

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$$\begin{aligned}
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&\geq \max\{0,\ a \| x_{00} \|^2 - \beta \| x \| (\| x_{01} \| + \| x_1 \|)\} \\
&+ \max\{0, c \| x_1 \| - A(\| x_{01} \|^2 + \| x_{00} \|^2), \\
&b \| x_{01} \| - \nu \| x_1 \| - \omega \| x_{00} \|^2\}.
\end{aligned} \tag{7}$$

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$$2^\circ. \quad b \parallel x_{01} \parallel - \nu \parallel x_1 \parallel - \omega \parallel x_{00} \parallel^2 >$$

Let $\varepsilon > 0$.
For each $x \in V_\varepsilon(0)$ : we obtain estimations:

$$1^\circ. \quad c \parallel x_1 \parallel - A(\parallel x_{01} \parallel^2 + \parallel x_{00} \parallel^2) \geq$$

$$\frac{A}{2}(\parallel x_{00} \parallel^2 + \parallel x_{01} \parallel^2 + \parallel x_1 \parallel^2);$$

$$2^\circ. \quad b \parallel x_{01} \parallel - \nu \parallel x_1 \parallel - \omega \parallel x_{00} \parallel^2 >$$

$$> \frac{\gamma}{4}(\parallel x_{01} \parallel^2 + \parallel x_{00} \parallel^2 + \parallel x_1 \parallel^2);$$

$$3^\circ. \quad a \parallel x_{00} \parallel^2 - \beta \parallel x \parallel (\parallel x_{01} \parallel + \parallel x_1 \parallel) >$$

$$3^\circ. \quad a \| x_{00} \|^2 - \beta \| x \| (\| x_{01} \| + \| x_1 \|) >$$

$$> K(\| x_{00} \|^2 + \| x_1 \|^2 + \| x_{01} \|^2)$$

$$3^\circ. \quad a \parallel x_{00} \parallel^2 - \beta \parallel x \parallel (\parallel x_{01} \parallel + \parallel x_1 \parallel) >$$

$$> K(\parallel x_{00} \parallel^2 + \parallel x_1 \parallel^2 + \parallel x_{01} \parallel^2)$$

where $\gamma > 0$,

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where $\gamma > 0, \ \ K > 0.$

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$$> K(\parallel x_{00} \parallel^2 + \parallel x_1 \parallel^2 + \parallel x_{01} \parallel^2)$$

where $\gamma > 0, \quad K > 0.$

Suppose:

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$$> K(\parallel x_{00} \parallel^2 + \parallel x_1 \parallel^2 + \parallel x_{01} \parallel^2)$$

where $\gamma > 0, \quad K > 0.$

Suppose: $\varepsilon \leq \min\{\frac{c}{2A}, \frac{cb}{4A\nu}, \frac{b}{4\gamma}\}, x \in V_\varepsilon(0).$

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Then inequality (3) is valid with $\delta = min\{\frac{A}{2}, \frac{\gamma}{4}, K\} > 0.$

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$$> K(\parallel x_{00} \parallel^2 + \parallel x_1 \parallel^2 + \parallel x_{01} \parallel^2)$$

where $\gamma > 0, \;\; K > 0.$

Suppose: $\varepsilon \leq \min\{\frac{c}{2A}, \frac{cb}{4A\nu}, \frac{b}{4\gamma}\}, x \in V_\varepsilon(0).$

Then inequality (3) is valid with $\delta = min\{\frac{A}{2}, \frac{\gamma}{4}, K\} > 0.$

Theorem 1 is proved.

# Abnormal Extrema in Non Linear Programming Problems: Optimality Conditions and Rigidity

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**Definition.**

The admissible solution $\hat{x} \in \mathbb{R}^n$ of problem (1) is called **regular**

**Definition.**

The admissible solution $\hat{x} \in \mathbb{R}^n$ of problem (1) is called **regular** if gradient vectors

$$\nabla h_i(\hat{x}),\ i = \overline{1, s}; \nabla g_j(\hat{x}),\ j \in J_A(\hat{x})$$

are linearly independent in this point.

**back**

$\hat{x}$ é regular $\Leftrightarrow Im \begin{pmatrix} \nabla H(\hat{x}) \\ \nabla G_{J_A}(\hat{x}) \end{pmatrix} = \mathbb{R}^{s+k}$, where $k = |J_A|$

$\Leftrightarrow \begin{pmatrix} \nabla H(\hat{x}) \\ \nabla G_{J_A}(\hat{x}) \end{pmatrix}$ is onto mapping.

**back**

there exists $\alpha > 0$ such that

$$\nabla^2_{xx}\mathcal{L}(\hat{x}, \hat{\lambda}_0, \hat{\lambda}, \hat{\mu})(\xi, \xi) \geq \ 2\alpha \parallel \xi \parallel^2, \ \forall \xi \in M. \tag{8}$$

**back**