A simple geometric condition for the normal cone intersection formula

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Outline

1 The Normal Cone Intersection Formula (NCF).

2 Standard Constraint Qualifications:
Interiority Conditions (IC).

3 Uses of (NCF).

4 Examples: (NCF) & no valid (IC).

5 Bounded linear regularity.

6 Technical Facts on Infimal Convolution and Conjugate Functions.

7 The Closure Condition (CC) .

8 Results and Applications

Normal Cone Intersection Formula: (NCF)

X real Banach space,
$C, D \subset X$ closed and convex.

We always have:

$$N_{D\cap C}(x) \supset N_D(x) + N_C(x), \;\; \forall\, x \in C \cap D$$

(NCF) holds when

$$N_{D\cap C}(x) = N_D(x) + N_C(x), \; \forall\, x \in C \cap D$$

The ncif states that the normal cone associated with $C \cap D$ is the sum of the normal cones assoc. with C and D, resp. The former is always bigger than the sum of the latter two. There are well-known conditions that guarranty the equality.

Usefulness of (NCF)

Given $f : X \to \mathbb{R} \cup \{+\infty\}$ proper convex and continuous at some point in $C \cap D$, consider the problem

$$\min\{f(x) \mid x \in C \cap D\}$$

$$\Updownarrow \quad \text{optimality conditions}$$

$$0 \in \partial f(x) + N_{C \cap D}(x)$$

If we have a convex function which is continuous at some point of $C \cap D$ we know that the solution of this problem is characterized by the inclusion. However, the significance of this description of the solution relies on the possibility of expressing $N_{C\cap D}(x)$ in terms of $N_C(x)$ and $N_D(x)$.

If (NCF) holds

The solution of problem

$$\min\{f(x) \mid x \in C \cap D\}$$

is characterized as $\Updownarrow$

$$0 \in \partial f(x) + N_C(x) + N_D(x)$$

Model for: Convex Constrained optimization, best approximation problems, etc.

So it is important and useful to establish conditions under which this formula holds. Let us recall the standard conditions for which the formula is known to hold.

Interiority Conditions: (IC)

$$int(C) \cap D \neq \emptyset$$

$$\Downarrow$$

$$0 \in core(C - D)$$

$$\Downarrow$$

(IC_) := $cone(C - D)$ closed subspace

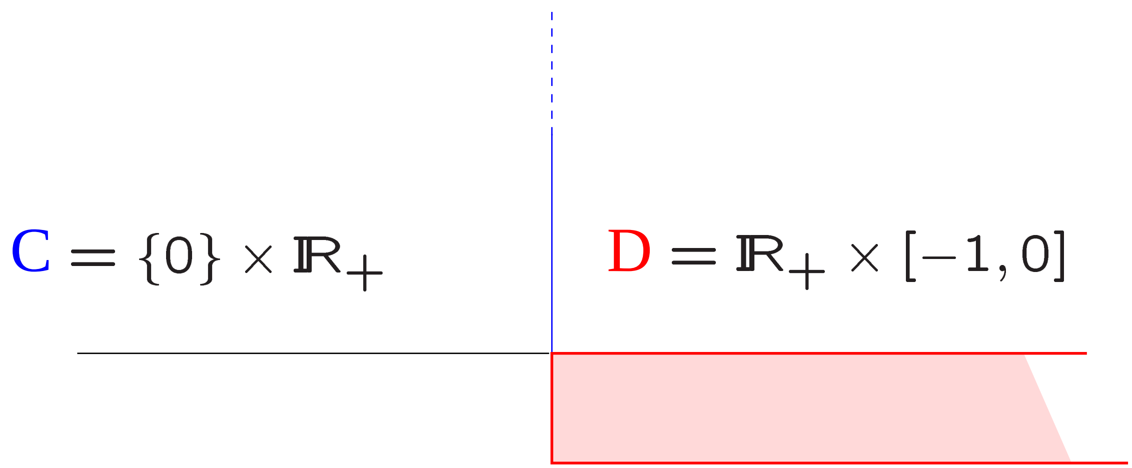
Under any of these (NCF) holds.

$0 \in core(A) \iff \forall x \in X\, \exists \varepsilon > 0$ such that $\lambda \in [-\varepsilon, \varepsilon]$ implies $\lambda x \in A$.

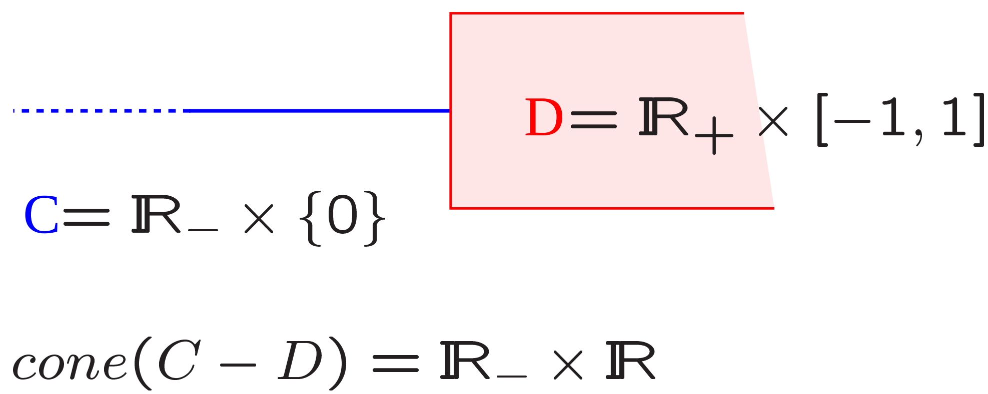
$D = \mathbb{R}_-$

$C = [0, 1]$

$$cone(C - D) = \mathbb{R}_+$$



$$cone(C - D) = \mathbb{R}_- \times \mathbb{R}_+$$



D= ℝ₊ × [−1, 1]
C= ℝ₋ × {0}
cone(C − D) = ℝ₋ × ℝ

In all these cases (IC_-) doesn't hold,

because $cone(C - D)$ not closed subspace.

However, (NCF) holds!

It has been simple to construct examples of C, D which don't satisfy (IC$_-$), but satisfy (NCF). For the case in which C, D are closed convex cones in a finite dimensional space, there exists an alternative condition, called Bounded Linear Regularity.

Case $C, D \subset \mathbb{R}^N$ closed convex cones

The closed and convex cones $\{C, D\}$ are Boundedly Linearly Regular when for every $S \subset \mathbb{R}^N$ bded there exists $\kappa_S > 0$ such that

$$d(x, C \cap D) \leq \kappa_s \max\{d(x, C), d(x, D)\}, \quad \forall x \in S$$

Fact 1: When $\{C, D\}$ are Boundedly Linearly Regular, then (NCF) holds. (Bauschke, Borwein, Li, 1999)

Fact 2: (NCF) may hold, with $\{C, D\}$ not Boundedly Linearly Regular. (Bauschke, Borwein, Tseng, 2000)

As we see by Fact 1, this BLR guarantees that (NCF) holds. However, it has been recently constructed a quite involved example (in R5), due to BBT, in which the condition BLR doesn't hold, while (NCF) still holds. So BLR is also stronger for this case. This justifies the quest of a tighter condition. In order to introduce this condition, we need to recall some technical facts.

Infimal Convolution

For $f, g : X \to \mathbb{R} \cup \{+\infty\}$ proper lsc. The infimal convolution:

$$f \oplus g(x) := \inf_{x_1+x_2=x}\{f(x_1) + g(x_2)\}$$

$f \oplus g$ is said to be exact when the inf is achieved $\forall\, x \in X$.

Fact 1: If $f \oplus g$ is exact, then

$$\text{Epi } (f \oplus g) = \text{Epi } f + \text{Epi } g$$

Fact 2: $cone(\text{dom f} - \text{dom g})$ closed subspace implies

$$f^* \oplus g^* \ \ \textbf{exact, and} \ \ f^* \oplus g^* = (f+g)^*$$

Some Notation

Fix $A \subset X$ closed and convex. Recall that $N_A = \partial \delta_A$, where

$$\delta_A(x) := \begin{cases} 0 & x \in A \\ +\infty & x \notin A. \end{cases}$$

Also, the Moreau-Fenchel conjugate of δ_A is

$$\delta_A^*(v) = \sup_{x \in A} \langle v, x \rangle =: \sigma_A(v).$$

Infimal Convolution of δ_C and δ_D

Take $f = \delta_C$ and $g = \delta_D$. Then

$$cone(\mathrm{dom\,f} - \mathrm{dom\,g}) = \mathrm{cone}(\mathrm{C} - \mathrm{D}).$$

If the latter is closed subspace then

$$f^* \oplus g^* = \sigma_C \oplus \sigma_D \text{ **exact**}$$

and

$$\sigma_C \oplus \sigma_D = (\delta_C + \delta_D)^* = \sigma_{C \cap D}$$

$$\mathrm{Epi}\,(\sigma_\mathrm{C} \oplus \sigma_\mathrm{D}) = \mathrm{Epi}\,\sigma_\mathrm{C} + \mathrm{Epi}\,\sigma_\mathrm{D}$$

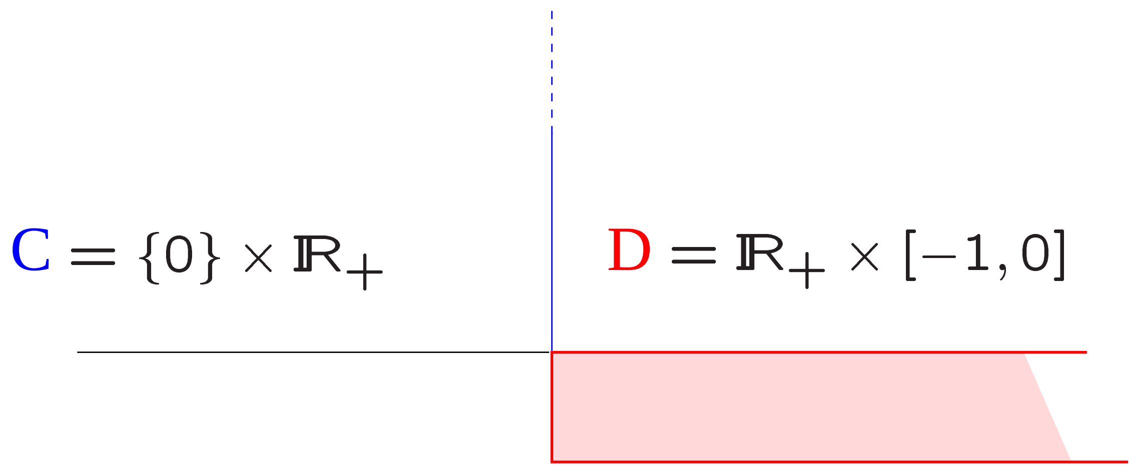
The Closure Condition: (CC)

Epi σ_C + Epi σ_D is weak* closed

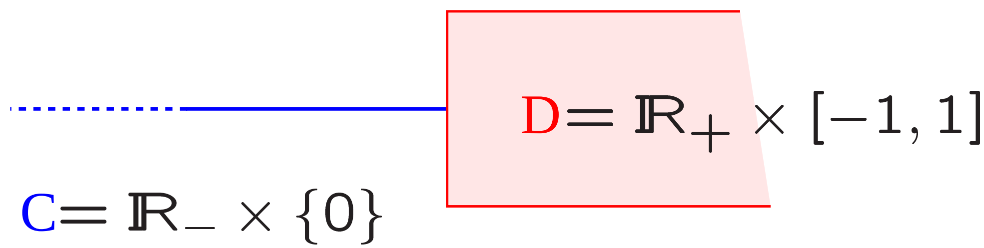
Fact: If $cone(C - D)$ closed subspace, then (CC) holds.

$D = \mathbb{R}_-$ $C = [0, 1]$

$$\mathrm{Epi}\,\sigma_C + \mathrm{Epi}\,\sigma_D = \mathbb{R} \times \mathbb{R}_+$$



$$\mathrm{Epi}\,\sigma_C + \mathrm{Epi}\,\sigma_D = \mathbb{R}^2 \times \mathbb{R}_+$$



$$\mathrm{Epi}\,\sigma_C + \mathrm{Epi}\,\sigma_D = \mathbb{R}^2 \times \mathbb{R}_+$$

Results-I

Fact 1: If $\mathrm{Epi}\,\sigma_C + \mathrm{Epi}\,\sigma_D$ is weak* closed, then (NCF) holds.

Fact 2: If $X = \mathbb{R}^N$ and C, D closed convex cones. Then

$$\boxed{\mathrm{Epi}\,\sigma_C + \mathrm{Epi}\,\sigma_D \text{ closed.}}$$

$$\Updownarrow$$

$$\boxed{\text{(NCF) holds.}}$$

Results-II

Dual cone of D
$D^+ = \{\theta \in X' : \theta(k) \geq 0, \forall k \in D\}$

Property: C, D closed convex cones, then

$$(C \cap D)^+ = \mathrm{cl}(C^+ \cap D^+)$$

Extension to arbitrary C, D such that $0 \in C \cap D$:

Corollary C, D closed convex such that $0 \in C \cap D$. If (Epi $\sigma_C +$ Epi σ_D) is weak* closed. Then

$$(C \cap D)^+ = C^+ + D^+.$$

Results-III

Fact 3: If $X = \mathbb{R}^N$ and C, D closed convex cones. If $\{C, D\}$ are Boundedly Linearly Regular, then Epi σ_C+Epi σ_D closed.